The many faces of Gravitational Wave Memory Effects

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Abstract

This lecture note is an overview of gravitational wave memory effects, delivered at the HolographyCL Farewell Meeting in Viña del Mar, Chile in January 2025.

1 Introduction

Gravitational wave memory (GWM)[1] refers to certain low-frequency features of gravitational waves (GW) predicted by general relativity. As we will show, Einstein equations entail a DC component in gravitational waves that is created in almost any radiative process, making GWM a universal property of GWs. Universality, together with observability are among the main motivations to study this topic. There are, however, other motivations such as potential astrophysical implications [].

An overview of GWM should consist, in my opinion, at least three topics:

1. **Generation.** Mathematical formulation of the memory produced in a dynamical process. We will address this from two different perpectives: the perturbative approach and the asymptotic analysis.

2. **Detection.** To find different experiments that capture persistent GW effects. The displacement effect, i.e. a permanent change in the physical distance between two free test masses is the classical example. But one should be creative and think of other setups.

3. **Interpretation.** Universality of GWM, like other universal effects in physics can be rooted to symmetries and conservation laws. Moreover, being a low-frequency feature of GW, it is fundamentally related to soft theorems. I classify these considerations as the interpretation of GWM. But actually, it is more that that, because this approach has motivated interesting developments in the first two topics, and provides a framework to study and understand low-frequency GW effects in a unified manner. Moreover, it can lead to important implications for holography and quantum gravity.

Due to limitations in time, we will restrict our attention to the first topic, *i.e.* the *generation* of memory. I will hopefully talk about *detection* in another school! The interpretation is already nicely covered by Strominger's lecture notes [1].

[1]Some people might use the abbreviation GME standing for gravitational memory effect. However, I would not waste a letter for "effect", because all physics is about effects. Moreover, with GWM, I want to insist that this is a subtopic in "gravitational wave theory".

Basic idea. GWM in linearized gravity is a direct consequence of the fact that the initial configuration of the system is different from the final configuration. Think about a classical scattering process, such as the hyperbolic encounter of two massive stars/black holes. The produced waveform has a DC offset (constant shift), because the momenta of the ingoing and outgoing particles are different, as we will see. More interestingly, general relativity (GR) implies an additional memory, called the *nonlinear memory*, which is sourced by the nonlinear interactions in the theory. This is also not surprising: the square of an oscillatory function, has a non-oscillatory part, *e.g.* , $\sin^2\varphi = \frac{1}{2} - \frac{1}{2}\cos(2\varphi)$. We will see these ideas in concrete setup in the following.

Early works. Gravitational wave memory was found in linearized gravity in the 70s and 80s in the works of Zel'dovich, Polnarev, Braginsky, Grishchuk, Thorne and others [2–5]. In 1991, Christodoulou, who had proven the stability of Minkowski together with Klainerman, used their machinary to make a novel prediction: the *nonlinear memory* effect [6]. Will and Wiseman showed that while the linear memory is the dominant effect for unbound systems (scattering binaries), it is the nonlinear memory is the dominant effect for bound systems (coalescing binaries). This is highly nontrivial, since the nonlinear memory is sourced by $O(G^2)$ effects, which is subleading with respect to the linear memory which appears at $O(G)$ in a perturbative expansion. The dominance of the nonlinear memory for bound orbits is due to the accumulation over the adiabatic evolution of the binary.

In 1992, Blanchet and Damour formulated memory effects from a completely different perspective: the perturbative post-Minkowskian formulation of GR, i.e. an expansion of the metric in powers of the Newton's constant[2]. They addressed not only the memory, but a more general class of *hereditary* GW effects, which depend on the entire past history of the system [7]. Examples of hereditary effects are the memory and *tail* effects. While memory refers to a final DC offset (constant shift) in the waveform, tail refers to a slow (power-law) decay in the waveform at late time.

While Blanchet and Damour found tail effects as a second-order effect in the PM theory over Minkowski spacetime, tail effects also appear in a linear perturbation over a curved background such as Schwarzschild or Kerr, which is thoroughly analyzed in black hole perturbation theory [8, 9]. In this context, tail effect arises because of a branch-cut in the Green function of the wave operator over the curved background. Finally, there are recent developments on tail effects using scattering amplitude methods. A careful comparison of these perspectives on low-frequency GW effects will be very fruitful.

2 Generation of GWM in perturbation theory

In this section, we will describe how GWM shows up in the post-Minkowskian formulation of GR, i.e. an expansion of the metric in powers of the Newton's constant. We will see that GWM appears at both linear and second order, and the corresponding memory is therefore called the linear and nonlinear memory effects.

We work with the gothic metric deviation defined as $h^{\mu\nu} = \sqrt{|g|}g^{\mu\nu} - \eta^{\mu\nu}$ and satisfying the de Donder (or harmonic) gauge condition $\partial_\mu h^{\mu\nu} = 0$ in Cartesian coordinates (t, x^i)

[2] Here is how the story began: one day, Thibault Damour brought a copy of the 1980 paper of Thorne [] to his new PhD student, Luc Blanchet, and asked him to extend it to the next order in the perturbation.

in which $ds^2 = -dt^2 + \delta_{ij}dx^i dx^j$. The Einstein field equations in harmonic coordinates read as

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4}|g|T^{\mu\nu} + \Lambda^{\mu\nu}(h, \partial h, \partial^2 h)\,, \tag{1}$$

where

$$\begin{aligned}\Lambda^{\alpha\beta} =& - h^{\mu\nu}\partial_\mu\partial_\nu h^{\alpha\beta} + \frac{1}{2}\partial^\alpha h_{\mu\nu}\partial^\beta h^{\mu\nu} - \frac{1}{4}\partial^\alpha h\partial^\beta h + \partial_\nu h^{\alpha\mu}\left(\partial^\nu h^\beta_\mu + \partial_\mu h^{\beta\nu}\right) \\ & - 2\partial^{(\alpha}h_{\mu\nu}\partial^\mu h^{\beta)\nu} + \eta^{\alpha\beta}\left[-\frac{1}{4}\partial_\tau h_{\mu\nu}\partial^\tau h^{\mu\nu} + \frac{1}{8}\partial_\mu h\partial^\mu h + \frac{1}{2}\partial_\mu h_{\nu\tau}\partial^\nu h^{\mu\tau}\right] + O(h^3)\end{aligned} \tag{2}$$

Note that $T^{\mu\nu} = T^{\mu\nu}[g;\psi]$ is a function of some matter source ψ (can be matter fields, or effective point-particles). For example, if the stress tensor describes a set of point particles, they follow geodesics, which crucially depends on the metric.

To solve Einstein equations, we consider a post-Minkowskian (PM) expansion of the metric in powers of the Newton's constant G,

$$h^{\mu\nu} = \sum_{n=1}^{+\infty} G^n h^{\mu\nu}_n\,. \tag{3}$$

Inserting this into (1), one can solve the equations iteratively, *i.e.* order by order in powers of G. We will first study the linear theory (1PM) and then second-order perturbations (2PM).

We should stress that while Eistein equations imply (1), the reverse is not true. It is the intersection of the solutions to (1) and the harmonic gauge condition $\partial_\mu h^{\mu\nu} = 0$ which corresponds to a solution of Einstein equations (assuming that the PM expansion is convergent). Therefore, (1) is called the relaxed Einstein equations. Moreover, taking the divergence of (1), we observe that the harmonic gauge condition implies that $\partial_\mu T^{\mu\nu}_{\text{eff}} = 0$, where $T^{\mu\nu}_{\text{eff}}$ is the rhs of (1). Therefore, harmonic gauge condition corresponds to the conservation of the effective gravitational stress tensor.

2.1 Linear memory in classical scattering

At leading order in the PM expansion $h^{\mu\nu} = Gh^{\mu\nu}_1$. From the definition, one can check that $h^{\mu\nu}_1$ is the trace-reversed metric perturbation, *i.e.* $g^{\mu\nu} = \eta^{\mu\nu} + G(h^{\mu\nu}_1 - \frac{1}{2}\eta^{\mu\nu}h_{(1)}) + \mathcal{O}(G^2)$ where $h_{(1)} = \eta_{\mu\nu}h^{\mu\nu}_{(1)}$. Inserting (3) in (1), one obtains at leading order

$$\Box h^{\mu\nu}_1 = \frac{16\pi}{c^4}T^{\mu\nu}[\eta;\psi]\,, \tag{4}$$

This can be solved given the Green function for the wave operator $\Box$, given by the retarded Green function integral in Cartesian coordinates

$$h^{\mu\nu}(x) = \frac{16\pi}{c^4}\int d^4x' G(x,x')T^{\mu\nu}(x')\,, \qquad G[x,y] = \frac{1}{2\pi}\theta(x^0 - y^0)\delta\left(|x-y|^2\right) \tag{5}$$

There are alternative ways to express the Green function, which can simplify the problem for specific sources. We consider the source to be a set of scattering pointlike particles, each described by the stress tensor

$$T^{\mu\nu}[g_{\mu\nu}, z^\mu(\tau)](x') = m\int \frac{u^\mu u^\nu}{\sqrt{-g}}\,\delta^{(4)}\left(x'^\alpha - z^\alpha(\tau)\right)\,d\tau, \tag{6}$$

Inserting (6) in (5), we find an expression involving $\int d\tau\delta\left(|x-z(\tau)|^2\right)$ which can be worked out by the identity $\delta(f(x))=\sum_i \delta(x-x_i)/f'(x_i)$ where x_i are the roots of the function f. We thus obtains the gravitational analogue of the Liénard–Wiechert potential

$$h_1^{\mu\nu}(x)=-\frac{4m}{c^2}\frac{u^\mu u^\nu}{u_\alpha\,(x-z)^\alpha}\Big|_{t=t_{\rm ret}}\,. \tag{7}$$

where $u^\mu=\frac{dz^\mu}{d\tau}$ is the velocity of the massive particle. At large distance, $r\equiv|\boldsymbol{x}|\gg|\boldsymbol{z}|$,

$$h_1^{\mu\nu}(t,r,\boldsymbol{n})=-\frac{4}{rc^2}\frac{p^\mu p^\nu}{k\cdot p}+\mathcal{O}(r^{-2})\,, \tag{8}$$

where the momentum $p^\mu=mu^\mu$ on the rhs is evaluated at time $u=t-r$, and $k^\mu=(1,n^i)$ is a null vector that specifies the location of the observer on the sphere.

The waveform, by definition is the $1/r$ term in the transverse-traceless projection of the metric, and the memory is the difference between the final and initial value of that quantity. For a scattering problem, the memory of the gravitational waveform, is therefore[3]

$$\Delta g_{ij}^{\rm TT}(\boldsymbol{n})=\lim_{t\to\infty}(h_{ij}(t)-h_{ij}(-t))^{\rm TT}=\frac{4G}{r}\sum_A\eta_A\left(\frac{p_A^i p_A^j}{k\cdot p_A}\right)^{\rm TT} \tag{9}$$

where $\eta_A=+1(-1)$ for outgoing (ingoing) particles and A sums over all constituents of the scattering. This is the result of Braginsky and Thorne [3]. Here TT refers to the "transverse-traceless" projection, using the projector

$$X_{ij}^{\rm TT}\equiv\perp_{ijkl}X_{kl}\,,\qquad \perp_{ijkl}=\tfrac{1}{2}\big(\perp_{ik}\perp_{jl}+\perp_{il}\perp_{jk}-\perp_{ij}\perp_{kl}\big),\qquad \perp_{ij}=\delta_{ij}-n_in_j\,.$$

Remark 1. As we mentioned before, solving the relaxed Einstein equations is not enough; one has to enfore the harmonic gauge condition, which corresponds to the conservation of the effective stress tensor. At linear order in the PM expansion, one has to impose $\partial_\mu T^{\mu\nu}[\eta,z(\tau)]=0$, which implies an equation for the matter source. In fact, it implies that the particles should move on the geodesics of the Minkowski metric, *i.e.* straight lines. But this would imply that the scattering is trivial. In other words, linearized Einstein equation cannot consistently describe the interaction between particles. Therefore, one has two options. The hard choice is to solve the problem at a higher PM order, which would provide with the worldline of the particles, as well as the radiated GWs to the desired order, see *e.g.* [10] and references therein. The easy option, which is what we have actually done here is to stick to the linearized theory, and solve the equations in two patches of spacetime, very early and very late in time, where the particles are far enough, so that they effectively follow straight lines. As we saw in (9), we have obtained the memory in terms of incoming and outgoing momenta, which are assumed to be given, while following option 1 would additionally provide the final momenta in terms of the initial momenta.

[3] The relative minus sign with respect to (8) is because $g_{\mu\nu}=\eta_{\mu\nu}-G(h^{(1)}_{\mu\nu}-\frac{1}{2}h^{(1)}\eta_{\mu\nu})+\mathcal{O}(G^2)$, where indices on $h^{(1)}_{\mu\nu}$ is lowered using the background metric $\eta_{\mu\nu}$.

Remark 2. The Weinberg soft theorem states that the scattering amplitude of N particles with an additional soft graviton emission takes the universal form

$$\mathcal{M}_{N+1}(p_1\cdots p_n;\omega q^\mu)=S_0\mathcal{M}_n(p_1\cdots p_n)\,,\qquad S_0=\sum_A\eta_A\frac{\epsilon_{\mu\nu}p_A^\mu p_A^\nu}{\omega\,q\cdot p_A}+\mathcal{O}(\log\omega) \tag{10}$$

While this is a result in the frequency domain, the memory is obtained in the time domain. Fourier transforming the $1/\omega$ in the soft factor, we get a step function in the time domain and this is the constant shift in the wave form, *i.e.* the memory. Moreover, from the saddle-point approximation, the integral over the graviton momenta localizes q^μ to the observer direction $k^\mu=(1,\boldsymbol{n})$. Finally, note an apparent tension: the above result for the memory is obtained only at linear order, while the soft theorem is valid at all orders in the perturbative expansion. One might worry that nonlinear corrections to the memory break the above nice correspondence between memory and soft factor. We will show in the following section that while there is a nonlinear correction to the memory, it still takes the same form as (9) for a scattering process. One only needs to sum over additional products that are produced through nonlinear interactions.

2.2 Second-order effect: non-linear memory

Let us now see what happens to the memory at second PM. From the linear analysis, we found that at large distance

$$h_1^{\mu\nu}=\frac{1}{r}\bar{h}^{\mu\nu}(u,\boldsymbol{n})+O(r^{-2}).$$

It will be convenient to decouple constant Coulombic contributions from radiative fields as follows:

$$\bar{h}^{00}=-4\big(M+n_iP^i\big)+z^{00}\,,\qquad \bar{h}^{0i}=-4P^i+z^{00}\,,\qquad \bar{h}^{ij}=z^{ij} \tag{11}$$

The effective stress tensor then takes the form

$$\Lambda^{\mu\nu}=\frac{1}{r^2}\Lambda_2^{\mu\nu}(u,\boldsymbol{n})+\mathcal{O}\left(r^{-3},G^3\right)\,,\qquad \Lambda_2^{\mu\nu}=4\big(M+n_iP_i\big)\ddot{z}^{\mu\nu}+k^\mu k^\nu\,\mathcal{F}(u,\boldsymbol{n})\,. \tag{12}$$

where $\dot{X}\equiv\partial X/\partial u$, and

$$\mathcal{F}=\frac{1}{2}\dot{z}^{\mu\nu}\dot{z}_{\mu\nu}-\frac{1}{4}\dot{z}^\mu{}_\mu\dot{z}^\nu{}_\nu\,. \tag{13}$$

In fact, it can be shown that $\mathcal{F}$ is flux of energy carried by GWs[4]

$$\frac{dE^{\rm GW}}{dud\Omega}=\frac{G}{16\pi}\mathcal{F}(u,\boldsymbol{n})+\mathcal{O}(G^2)\,. \tag{14}$$

Inserting (12) on the rhs of (1) and solving for h_2, one finds [7]

$$h_2^{\mu\nu}=h_2^{\mu\nu}\Big|_{\rm mem}+h_2^{\mu\nu}\Big|_{\rm tail}+\text{instantaneous terms}\,. \tag{15}$$

[4]This can be shown either by using the Landau-Lifshitz pseudo tensor, or by showing that it matches with the well-known Bondi energy flux.

where

$$h_{2ij}^{\rm TT}\Big|_{\rm mem} = \frac{1}{r}\left(\sum_{\ell\geqslant 2}\frac{2n_{L-2}}{(\ell+1)(\ell+2)}\int_{-\infty}^{u} dv\, \mathcal{F}_{ijL-2}(v)\right)^{\rm TT}\,, \tag{16}$$

where $\mathcal{F}_L$ denotes the decomposition of $\mathcal{F}$ in STF harmonics

$$\mathcal{F}(u,\boldsymbol{n}) = \sum_{\ell=0}^{+\infty} n_L\,\mathcal{F}_L(u) \quad\Leftrightarrow\quad \mathcal{F}_L(u) = \frac{(2\ell+1)!!}{\ell!}\int\frac{d\Omega}{4\pi}\, n_{\langle L\rangle}\,\mathcal{F}(u,\boldsymbol{n})\,. \tag{17}$$

Eq. (16) is the nonlinear GWM to second PM order. The memory is clearly sourced by the flux of energy through gravitational radiation at leading PM order.

Lemma. For unit vectors $\boldsymbol{n},\boldsymbol{n}'$ representing two points on the sphere, the following identity holds (see [11] for the proof)

$$\left[\frac{n'_i n'_j}{1-\boldsymbol{n}\cdot\boldsymbol{n}'}\right]^{\rm TT} = \left[2\sum_{\ell=2}^{+\infty}\frac{(2\ell+1)!!}{(\ell+2)!}\, n_{L-2}\,\hat{n}'_{ijL-2}\right]^{\rm TT} \tag{18}$$

As a result of this lemma, one finds that

$$h_{2ij}{}^{\rm TT}\Big|_{\rm mem}(u,\boldsymbol{n}) = 4\perp_{ijkl}\int_{-\infty}^{u} du'\int d\Omega'\,\frac{n'^k n'^l}{1-\boldsymbol{n}\cdot\boldsymbol{n}'}\,\frac{dE^{\rm GW}}{d\Omega'}(u',\boldsymbol{n}')\,, \tag{19}$$

Thorne [5] pointed out that the nonlinear memory (19) is of the form of the linear memory (9) once we replace the particle momentum with that of a gravitons generated in the scattering process. One just needs to associate the momentum vector $p^\mu = \frac{dE^{\rm GW}}{d\Omega}(1,\boldsymbol{n})$ to the emitted graviton[5]. Therefore, the full memory to second order is still of the form (9), but one has to sum over outgoing hard gravitons that are produced due to nonlinear interactions.

2.3 Detour on tail effects

The second term in (15) reads

$$h_2^{\mu\nu}\Big|_{\rm tail} = \frac{2(M+n_iP_i)}{r}\int_{-\infty}^{u} dv\,\ln\left(\frac{u-v}{2b_0}\right)\ddot{z}^{\mu\nu}(v,\boldsymbol{n}) + \mathcal{O}\left(\frac{1}{r^2}\right)\,. \tag{20}$$

where b_0 is an irrelevant length scale to make the argument of the logarithm dimensionless. Eq.(20) is called the tail effect and leads to a slow decay of the waveform. It results from the coupling between the ADM conserved quantities M, P^i and the gravitational perturbation $z^{\mu\nu}$, *i.e.* , the fact that the perturbation is propagating on a curved background. The slow decay is because the graviton can back-scatter from the curved background, and therefore arrive much later at null infinity.

[5]Thorne writes in [5]:"The main purpose of this paper is to show that Christodoulou's effect, in fact, is included in the general expression (1) for the memory, but the author had missed it there due to his obtuseness."

Let us assume that linear waveform has compact support in time, i.e. $\dot{z}^{\mu\nu} = 0$ outside the interval (u_0, u_f). Then, an integration by part reveals

$$h_2^{\mu\nu}\Big|_{\rm tail} = \frac{2(M + n_i P_i)}{r} \int_{u_0}^{u_f} dv \, \frac{\dot{z}^{\mu\nu}(v, \boldsymbol{n})}{u - v} + \mathcal{O}\left(\frac{1}{r^2}\right) . \tag{21}$$

The behavior of this expression at late-time $u \gg u_f$ can be obtained by expanding $\frac{1}{u-u'} = \frac{1}{u} \sum_{n=0}^{\infty} \left(\frac{u'}{u}\right)^n$ in (21), and we find

$$h_2^{\mu\nu}\Big|_{\rm tail} = \frac{2(M + n_i P_i)}{r} \sum_{n=1}^{\infty} \frac{\mathcal{M}_n\left(\dot{z}_1^{\mu\nu}\right)}{u^n} \tag{22}$$

where $\mathcal{M}_n(f) = \int_{u_0}^{u_f} du u^{n-1} f(u)$ is the Melling transform of the linear waveform. Therefore, the leading-order $n = 1$ tail effect decays as $1/u$ and is proportional to the linear memory $\Delta h_1^{\mu\nu}$.

3 GWM from the asymptotic analysis

An alternative approach to derive memory effects is to use the asymptotic analysis. We saw in the previous section that the memory was dependent merely on the ingoing and outgoing state of the system, and not on the intermediate dynamics. We will make this more rigorous in this section.

3.1 Asymptotics at null infinity

We saw that the harmonic coordinate system provides a powerful setup to study Einstein equations perturbatively. However, they are not particularly suited to describe the asymptotics. The reason is that the coordinate $u = t - r$ diverges logarithmically from the physical outgoing lightcones.

Instead, in Bondi coordinates, we let the radiative fields define the coordinate system for us. To see how it works, consider a source emitting EM radiation. Using the Eikonal method, we write the gauge field $A_\mu = a_\mu e^{i\psi/\epsilon}$, where $\epsilon \ll 1$ indicates that the phase is a fast variable, while the amplitude a_μ is a slow variable. Inserting this into Maxwell equations outside the source $\nabla_\mu F^{\mu\nu} = 0$, we find at $O(\epsilon^{-2})$ that

$$g^{\mu\nu} k_\mu k_\nu = 0 \,, \qquad k_\mu \equiv \partial_\mu \psi \tag{23}$$

which explains that EM wavefronts described by $\psi = const$ hypersurfaces are null and therefore the phase ψ provides a null foliation of spacetime. By construction $k^\nu \nabla_\nu k_\mu = k^\nu \nabla_\nu \nabla_\mu \psi = k^\nu \nabla_\mu \nabla_\nu \psi = k^\nu \nabla_\mu k_\nu = 0$, which implies that k^μ is the generator of null geodesics along null hypersurfaces[6]. So why not take the scalar ψ as a coordinate variable? We therefore define the Bondi time $u = \psi$, which is also called the eikonal phase as it solves the eikonal equation $g^{\mu\nu} \partial_\mu u \partial_\nu u = 0$. In this coordinate system $k_\mu = (1, \mathbf{0})$, and the eikonal equation implies $g^{uu} = 0$. We further define the angular coordinates θ^A to be

[6] At $O(\epsilon^{-1})$, we find that the polarization obeys $k^\mu \nabla_\mu f^\nu = 0$, where $f^\nu \equiv a^\nu/|a|$, i.e. that the polarization is parallel transported along the ray.

constant along the generators $k^\mu \equiv g^{\mu\nu}k_\nu$ of the null cones, implying that $k^\mu\partial_\mu\theta^A = 0$. These imply that

$$g^{uu} = 0 = g^{uA} \tag{24}$$

This is the (partial) Bondi gauge conditions. The general form of the metric in the radiative gauge reads

$$ds^2 = -e^{2\beta}du(Fdu - 2dr) + r^2\gamma_{AB}(d\theta^A - U^A du)(d\theta^B - U^B du) \tag{25}$$

We need to complete the gauge by imposing a condition on the parameter of the outgoing null curves generated by k^μ. Two important options are

- Define r to be the affine parameter along the outgoing null curves implying that $g_{ur} = -1$ (Newman-Unti gauge)
- Define r to be the areal distance from the origin, implying that $\det(\gamma_{AB}) = \det(q_{AB})$, where q_{AB} is the round metric on the unit sphere. (Bondi gauge)

With respect to the round metric $q_{AB}(\theta^A)$ on the sphere with Ricci scalar $R[q] = 2$, we can decompose γ_{AB} into a trace and a traceless part

$$\gamma_{AB} = \Omega q_{AB} + h_{AB}\,, \qquad q^{AB}h_{AB} = 0 \tag{26}$$

From this, we find $\det(\gamma) = \det(q)\left(\Omega^2 - \frac{1}{2}h_{AB}h^{AB}\right)$. Imposing the Bondi gauge condition implies that $\Omega = \sqrt{1 + h_{AB}h^{AB}/2}$, and thus the inverse γ^{AB} is given simply by $\gamma^{AB} = \Omega q^{AB} - h^{AB}$, where indices on the sphere are raised and lowered with the metric q_{AB} and its inverse q^{AB}. Now, perform an asymptotic expansion of the metric

$$h_{AB} = \frac{1}{r}C_{AB} + \sum_{n=3}^{\infty}\frac{1}{r^n}E^{(n)}_{AB}\,, \tag{27}$$

insert it back in the metric and solve Einstein equations asymptotically. The result is

$$\begin{aligned} ds^2 = &-(1 - 2m/r)du^2 - 2dudr(1 + O(r^{-2})) + (D_B C^{AB} + O(r^{-1}))dud\theta^A && (28)\\ &+ (r^2 q_{AB} + rC_{AB} + O(r^{-1}))d\theta^A d\theta^B\,. && (29)\end{aligned}$$

with a constrain between the mass aspect $m(u, \theta^A)$ and the Bondi shear $C_{AB}(u, \theta^A)$

$$\dot{m} = \frac{1}{4}D_A D_B \dot{C}^{AB} - \frac{1}{8}\dot{C}^{AB}\dot{C}_{AB} \tag{30}$$

where overdot refers to derivation wrt u. The Bondi shear C_{AB} is the leading term in the TT component of the metric, written in an angular basis $C_{AB} = r{e_A}^i{e_B}^j g^{\mathrm{TT}}_{ij}$. Therefore, it represents the gravitational waveform. Also, q_{AB}, D_A are respectively the round metric on the sphere and the corresponding covariant derivative. Integrating (30) over time and rearranging implies

$$D_A D_B \Delta C^{AB} = 4\Delta m + \frac{1}{2}\int_{-\infty}^{\infty} du\,\dot{C}^{AB}\dot{C}_{AB} \tag{31}$$

We observe that we have an expression for the memory in terms of the angular energy flux, and the total change in the Bondi mass aspect. However, there are two problem here

Q1: If I know C_{AB} as a function of time, I can directly compute the memory ΔC^{AB}. Why should I do this mess to derive the memory?

Q2: The Bondi mass is an unknown of the Bondi analysis. How informative is this equation?

Answer to Q1. The answer is best provided in the Fourier space. To simplify the notation, we note that any STF tensor on the sphere has only two d.o.f, and thus can be traded with a complex scalar. Given a null dyad $m^A, \bar{m}^A$ on the sphere, with $m \cdot \bar{m} = 1$, we define

$$C \equiv \bar{m}^A \bar{m}^B C_{AB} \quad \Leftrightarrow \quad C_{AB} = m_A m_B C + \bar{m}_A \bar{m}_B \bar{C} \tag{32}$$

The complex functions C and its complex conjugate $\bar{C}$ represent right and left-handed polarization of the radiation field respectively. In Fourier form,

$$C(u,n) = i \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} C(\omega,n)\, e^{-i\omega u} \tag{33}$$

Therefore

$$\Delta C(n) = \int_{-\infty}^{\infty} du \dot{C}_{AB} = \lim_{\omega \to 0} \omega C(\omega, n) \tag{34}$$

$$\frac{1}{2}\int_{-\infty}^{\infty} du \dot{C}^{AB} \dot{C}_{AB} = \int_{0}^{\infty} d\omega \omega^2 C^*(\omega,n) C(\omega,n) \tag{35}$$

From these two equations, we learn that the memory is directly related to zero frequency modes, while the flux contains all the frequency spectrum (hard modes). In particular, soft modes' contribution to energy flux is vanishingly small. Note also that a nonzero memory corresponds to a soft pole in the low-frequency expansion of the shear.

Another way to address this issue is to think about the PM expansion of this equation. Since $C_{AB} \sim O(G)$, the linear term is $O(G)$, while the flux term is $O(G^2)$. Therefore, at leading order, we can write

$$D_A D_B \Delta C^{AB}_{(1)} = 4 \Delta m_{(1)} \tag{36}$$

This reproduces the linear memory (9) for the scattering process, after replacing $\Delta m_{(1)}$ with its value, as discussed below. Moreover, one can argue that Δm is $O(G)$ exact, *i.e.* $\Delta m = G \Delta m_{(1)}$. Therefore, at second order, we find

$$D_A D_B \Delta C^{AB}_{(2)} = \frac{1}{2} \int_{-\infty}^{\infty} du\, \dot{C}^{AB}_{(1)} \dot{C}^{(1)}_{AB} \,. \tag{37}$$

This is actually a rewriting of (16).

Both equations take the form $D_A D_B X^{AB} = F$, where X_{AB} is an STF tensor. This equation can be solved as follows. First expand the STF tensor into its electric and magnetic components as

$$X_{AB} = D_{AB} X^+ + \epsilon_A{}^C D_{BC} X^- \,, \qquad D_{AB} \equiv D_A D_B - \frac{1}{2} q_{AB} D^2 \tag{38}$$

where $D^2 = q^{AB}D_A D_B$ is the Laplacian on the sphere. Inserting back into the equation, and some commutations of the derivatives implies

$$\frac{1}{2}D^2(D^2+2)X^+ = F \tag{39}$$

which can be solved as $X^+ = \int_{S^2} d^2\Omega'\, G(\boldsymbol{n},\boldsymbol{n}')\, F(\boldsymbol{n}')$ with the Green function

$$G(\boldsymbol{n},\boldsymbol{n}') = \frac{1}{4\pi}x\ln x, \qquad x = 1-\boldsymbol{n}\cdot\boldsymbol{n}' \tag{40}$$

that solves

$$\frac{1}{2}D^2\left(D^2+2\right)G\left(\boldsymbol{n},\boldsymbol{n}'\right) = \frac{1}{\sqrt{2}}\delta^2\left(\theta,\theta'\right) - \frac{1}{4\pi}\left(1+3\boldsymbol{n}\cdot\boldsymbol{n}'\right). \tag{41}$$

Once X^+ is solved, one can insert it back in (38) to find X_{AB}. Of course, this equation does not fix the magnetic part X^-. In our problem, the magnetic memory ΔC^- is assumed to vanish, which is consistent with a large space of solutions.

Answer to Q2: Indeed, the asymptotic analysis at null infinity does not provide any information about Δm in terms of the source. To address Q2, we have to match the analysis at null infinity to an asymptotic analysis at future timelike infinity. We will do this in the following.

Minimal approach. Let us focus on a scattering problem. We assume that the system is free/non-interacting in the far past and far future. A free particle with mass M and velocity $\boldsymbol{v}$ is described by a boosted Schwarzschild solution. Transforming that into Bondi form, one obtains

$$m = \frac{M}{\gamma^3\left(1-\boldsymbol{v}\cdot\boldsymbol{n}\right)^3}, \qquad \gamma(v) = (1-v^2)^{-1/2}, \qquad \vec{n}\cdot\vec{n} = 1 \tag{42}$$

Alternatively, one can start from Schwarzschild solution in Bondi coordinates, which corresponds to a constant mass $m = M$. Then a boosted Schwarzschild is obtained by acting a boost on this constant mass aspect. We know that infinitesimal Lorentz transformations are described by a vector field on the celestial sphere

$$\xi = Y^A\partial_A = (q^{AB}D_B\phi + \varepsilon^{AB}D_B\psi)\partial_A\,, \tag{43}$$

where $\phi = v^i n_i$ represents a boost with velocity $\boldsymbol{v}$ and $\psi = \omega^i n_i$ with an infinitesimal rotation $\boldsymbol{\omega}$ is the infinitesimal angular rotation vector. For the Schwarzschild solution which corresponds to $m = M$ and $C_{AB} = 0$, the transformation of the mass aspect is given by (see e.g. (2.23) of [12])

$$\delta_Y M = \frac{3}{2}MD_AY^A = \frac{3}{2}MD^2\phi = -3M\boldsymbol{v}\cdot\boldsymbol{n} \tag{44}$$

Indeed, (42) is the finite version of this infinitesimal transformation.

Since the system is asymptotically free at $u\to\pm\infty$, one can superpose (42) and find that

$$m = \sum_{A\in\text{outgoing}} \frac{M_A}{\gamma_A^3\left(1-\boldsymbol{v}_A\cdot\boldsymbol{n}\right)^3} \tag{45}$$

This together with a similar analysis in the far past reveals and equation for Δm in terms of outgoing and ingoing massive particles

$$\Delta m = \sum_A \eta_A \frac{M_A}{\gamma_A^3 \left(1 - \boldsymbol{v}_A \cdot \boldsymbol{n}\right)^3} \tag{46}$$

where $\eta_A = 1(-1)$ for outgoing (ingoing) particles.

3.2 Matching to future infinity

The more rigorous approach consists of matching our analysis at null infinity to an analysis of future timelike infinity i^+, on which the asymptotic state of massive degrees of freedom live. This is what we do in the following.

On the Penrose diagram, future infinity i^+ is singular. To study that region of spacetime, we have to *resolve the singularity*, by finding a coordinate system adapted to that region of spacetime. In 1982, Beig and Schmidt [13] found a beautiful description of spatial infinity, which can be applied to future timelike infinity with minimal changes.

Define the hyperbolic coordinate system (τ, ϕ^a) with $a = 1, 2, 3$ and impose the gauge condition $g_{\tau a} = 0$. Then we make an asymptotic expansion in the $\tau \to \infty$ limit to obtain

$$ds^2 = -\big(1 + 2\sigma/\tau + O(\tau^{-2})\big)d\tau^2 + \tau^2\Big(h_{ab} + \frac{1}{\tau}(k_{ab} - 2k h_{ab}) + O(\log\tau/\tau^2)\Big)d\phi^a d\phi^b \tag{47}$$

Asymptotic analysis. We consider Einstein equations with a set of pointlike particles as source, like the one in (6).

At leading order, Einstein equations imply that the three-dimensional metric h_{ab} has to solve $R_{ab} = 2h_{ab}$, i.e. Euclidean Einstein equation with cosmological constant $\Lambda = 1$. A particular solution is the hyperbolic metric in global coordinates (ρ, θ^A)

$$h_{ab}d\phi^a d\phi^b = d\rho^2 + \sinh^2\rho \,(q_{AB}\, d\theta^A d\theta^B) \tag{48}$$

We define an asymptotically flat spacetime at timelike infinity to be a solution of Einstein equations that asymptotes to h_{ab} in (48), as a fixed structure.

At subleading order, one obtains an equation for σ and another one for k_{ab}. The former reads

$$(D^2 - 3)\sigma = \sum_{n=1}^{N} 4\pi M_n \frac{\delta^{(3)}\left(\phi - \phi_n\right)}{\sqrt{h}} \tag{49}$$

which can be solved by a suitable Green function of the hyperbolic Laplace equation (see appendix A of [14] or appendix B of [15]). For a proper matching to $\mathcal{I}^+$, we choose the Green function such that $\lim_{\rho\to\infty}\sigma = 0$. With this condition, the unique solution to (49) is [15]

$$\sigma = \sum_{n=1}^{N} M_n \left(2\chi_n - \frac{2\chi_n^2 - 1}{\sqrt{\chi_n^2 - 1}}\right), \quad \chi_n = \gamma_n\left(\cosh\rho - v_n^i n_i \sinh\rho\right), \tag{50}$$

with the asymptotic behavior

$$\sigma \sim \sum_{n=1}^{N} \frac{-2M_n}{\gamma_n^3 \left(1 - v_n^i n_i\right)^3} e^{-3\rho}, \quad \rho \to \infty \tag{51}$$

We thus impose as a matching condition between timelike and null infinity that

$$\lim_{u \to \infty} m(u, \boldsymbol{n}) = \lim_{\rho \to \infty} e^{3\rho} \sigma(\rho \boldsymbol{n}) \tag{52}$$

This reproduces the results of the previous section. It can also be derived by imposing that the charge corresponding to supertranslations match at the intersection of the boundaries of future timelike and null infinity. Note that a similar procedure at past timelike infinity i^- and assuming no-incoming radiation at $\mathcal{I}^-$ reveals (46).

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