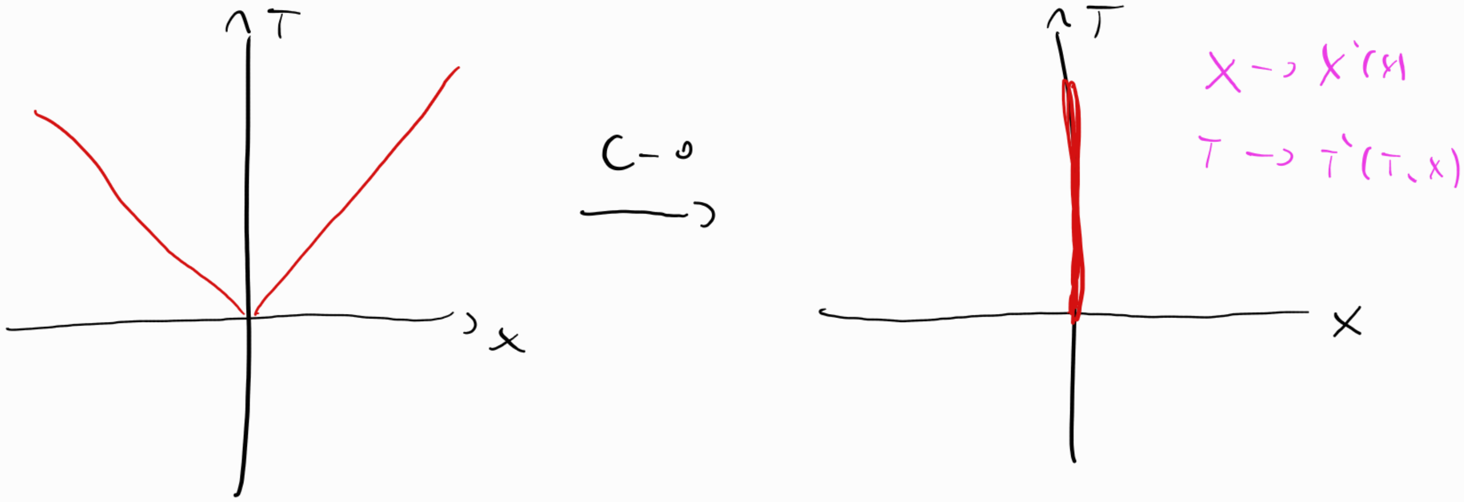
Carrollian dynamics appear as the ultrarelativistic limit of a relativistic theory ($c \to 0$)



The space becomes absolute and movement is then forbidden (ultralocal)

Although exotic, Carrollian dynamics and Carrollian geometry have been shown to have interesting applications in physics

1) Carrollian geometries from:

- Ultrarelativistic limit of pseudo-Riemannian manifold

- null hypersurfaces of Lorentzian spacetimes
  - Black hole event horizon
  - null infinity

2) $BMS_4 \simeq CCarr(3)$

asymptotic symmetry of gravity

3) Possible Ricci-flat / CCFT

$$\begin{tikzcd}
\text{AAdS} \arrow[r, "r \to \infty"] \arrow[d, "\Lambda \to 0"'] & \text{CFT (at timelike boundary)} \arrow[d, "c \to 0"] \\
\text{AFlat} \arrow[r, "r \to \infty"] & \text{Carrollian CFT (null boundary)}
\end{tikzcd}$$

4) More applications: Fractons, Tensionless strings,

etc.

The purpose of this mini course is to give an introduction to Carrollian dynamics. The outline is the following:

Outline

1) • Carroll Symmetries

• Carroll geometry:
  - From $c \to 0$ limit
  - null hypersurface
  - (Conformal) isometries

2) Carrollian Dynamics: - From symmetries
  - From the $c \to 0$ limit

- (non)-conserved Noether currents

3) Some application in Gravity :

- BH event horizon dynamics as Carrollian fluids

- Holographic reconstruction of Ricci-flat spacetimes

Lorentz

The complete set of symmetries of special relativity is given by the Poincaré group. The latter represents all the transformations that leave the Minkowski spacetime invariant, namely, the isometries of flat spacetime.

These transformations are the following:

1) Space-time translation $P_\mu = \partial_\mu$

2) Spatial rotation

3) Boosts

$$\left.\begin{matrix} \text{2) Spatial rotation} \\ \text{3) Boosts} \end{matrix}\right\} \quad J_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu \quad \text{(Lorentz transformation)}$$

Infinitesimally, the latter implies

$$\mathcal{L}_\xi \eta_{\mu\nu} = 0$$

$$\Rightarrow \partial_\mu \xi_\nu + \partial_\nu \xi_\mu = 0$$

$$\eta_{\mu\nu} = -c^2 dT^2 + \delta_{ij}\, dx^i dx^j$$

The latter implies That ξ^β depends on x^μ at most linearly!!

$$\rightarrow \boxed{\xi^\mu = \Omega^\mu{}_\nu X^\nu + a^\mu}$$, $\Omega^\beta{}_\alpha$ a constant tensor, and a^β a constant vector.

The Killing equation implies

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \partial_\mu(\Omega_{\nu\alpha} X^\alpha + a_\nu) + \partial_\nu(\Omega_{\mu\alpha} X^\alpha + a_\mu)$$

$$= \Omega_{\nu\alpha}\delta^\alpha_\mu + \Omega_{\mu\alpha}\delta^\alpha_\nu = 0$$

$$\rightarrow \Omega_{\nu\mu} = -\Omega_{\mu\nu} \rightarrow \Omega_{\mu\nu} \text{ antisimmetric}$$

Therefore

$$\xi^\mu \partial_\mu = (\Omega^\mu{}_\nu X^\nu + a^\mu)\partial_\mu$$

$\Omega^\mu{}_\nu X^\nu \equiv$ Rotation in $SO(d)$ + Boosts (Lorentz group)

$a^\mu \equiv$ Space-time translation

In Cartesian coordinates the generator of these transformations are

- $P_\mu = \partial_\mu$

- $J_{\mu\nu} = x_\mu \partial_\nu - \partial_\nu x_\mu$

The latter form the Poincaré algebra $ISO(d,1)$

Contraction of Poincaré

We can take diferent contraction of the Poincaré algebra, depending on what limit we take on the speed of light c.

To Galileo

This is obtain on the $c \to \infty$ limit of the Poincaré algebra (non-relativistic limit).

For instance, we can take the generators

$$P_0 = \frac{1}{c}\partial_T \quad , \quad P_i = \partial_i$$

$$J_{0i} = cT\partial_i - \frac{x_i}{c}\partial_T \quad , \quad J_{ij} = x_i\partial_j - \partial_j x_i$$

We can define the Galilean generators as the following limits:

$$\boxed{H = \lim_{c\to\infty} c P_0 = \partial_T} \,, \quad \boxed{P_i = \partial_i} \,,$$

$$\boxed{G_i = \lim_{c\to\infty} \frac{1}{c} J_{0i} = T\partial_i} \,, \quad \boxed{J_{ij} = x_i\partial_j - x_j\partial_i} \,.$$

These are the generators of all the transformations that keep a flat Newton-Cartan structure invariant.

Infinitesimally, these transformations are

$$T' = T + a \,, \quad a \text{ cte}$$

$$x'^i = x^i + \Omega_j{}^i x^j + T V^i + X^i$$

The latter form the Galilean algebra

$$[H, G_i] = P_i \,, \quad [G_i, P_j] = 0$$

To Carroll

The Carrollian algebra is obtained through the ultrarelativistic limit of the Poincaré algebra ($c \to 0$). Its generators are

$$\boxed{H = \lim_{c \to 0} c P_0 = \partial_T} \;, \quad \boxed{P_i = \partial_i}$$

$$\boxed{C_i = \lim_{c \to 0} c J_{0i} = x_i \partial_T} \;, \quad J_{ij} = x_i \partial_j - x_j \partial_i$$

Which form the Carrollian algebra

$$[H, C_i] = 0 \;, \quad [C_i, P_j] = -H \delta_{ij}$$

The Hamiltonian becomes a central element of the algebra.

The latter are the generators of all the Carrollian transformations that form the group of isometries of a flat Carroll spacetime.

Infinitesimally they act on

$$T' = T + a + B_i X^i$$

$$X' = X + \Omega_j{}^i x^j + X^i$$

$$\Rightarrow \underline{X \to X'(x)}$$

Carroll Spacetime

Let first starts with a $d+1$ pseudo-Riemannian manifold M with metric

$$ds^2 = -c^2(\Omega dT - b_i dx^i)^2 + a_{ij} dx^i dx^j$$

This choice of gauge is such that the dependence on c is explicit and is stable under Carrollian diffeos

$$T' = T'(T, x) \,, \quad \underbrace{x' = x'(x)}$$

absolute space!!

with Jacobian

$$\boxed{J(T,x) = \frac{\partial T'}{\partial T}, \quad j_i(T,x) = \frac{\partial T'}{\partial x^i}, \quad J^i{}_j = \frac{\partial x'^i}{\partial x^j}}$$

Providing the appropriate transformation.

For the function that parameterized the pseudo-Riemannian metric we have

- $\Omega(T,x)$ Transforms as a Carroll scalar

$$\Omega' = \frac{\Omega}{J}$$

- The d-dimensional metric a_{ij} transforms as a Carroll tensor

$$a'_{ij} = (J^{-1})_i{}^k (J^{-1})_j{}^l a_{kl}$$

- b_i is a connection

$$b'_i = (J^{-1})_i{}^k b_k + \frac{\Omega}{J}(J^{-1})_i{}^k j_k$$

If we now take the ultrarelativistic limit $c \to 0$, our geometry becomes a manifold $C \equiv \mathbb{R} \times \Sigma$ with degenerate metric

$$\boxed{ds^2 = 0 \cdot dT^2 + a_{ij}\, dx^i dx^j}$$

a_{ij} the metric of Σ with inverse a^{ij}

The time direction $\vec{v} = \frac{1}{\Omega} \partial_T$ defines the kernel, namely, it satisfies

$$g(\vec{v}) = 0.$$

$\vec{v}$ is typically called Carroll vector and it has a dual one-form

$$\underbrace{\tau}_{\text{Clock form}} = \Omega\, dT - b_i\, dx^i$$

$b_i dx^i$ → Ehresmann connection (makes possible the decomposition $TM \equiv V \oplus H$)

such that $\vec{v}(\tau) = 1$

The latter provides a decomposition of the vector in a vertical part on a horizontal part.

$\Rightarrow$ $TC = V \oplus H \longrightarrow$ $X \in TM$ that $X(T) = 0$

vector align with ∂_T

Here, vectors are decomposed as

$$W = W^0(T,x) E + W^i(T,x) E_i$$

$$= \underbrace{\frac{W^0}{\Omega} \partial_T}_{\in V} + \underbrace{W^i \hat{\partial}_i}_{\in H} \quad , \text{ with } \hat{\partial}_i := \partial_i + \frac{b_i}{\Omega} \partial_T$$

While forms are decomposed in terms of the co-frames as

$$w = w_0(T,x)\, e + w_i(T,x)\, e^i$$

with

$$\boxed{e = \Omega\, dT - b_i\, dx^i \ , \ e^i = dx^i}$$

Under Carrollian diffeos, they transform on

$$E' = E \ , \quad E'_i = (J^{-1})^j{}_i \, E_j$$

$$e' = e \ , \quad e'^i = J^i{}_j \, e^j$$

From the horizontal perspective

W^0 and W_0 are Carrollian scalars

While

W^i and W_i are Carrollian vector or one-form

Carroll Structures

Our Carroll spacetime is given by:

a $d+1$ manifold $C \equiv \overbrace{\mathbb{R} \times \Sigma}^{\text{Fiber Bundle}}$ equipped with

1) a degenerate metric

$$ds^2 = 0 \cdot dT^2 + a_{ij}\, dx^i\, dx^j$$

metric of S

Weak Carroll structure

2) Carroll vector $\vec{v} = \frac{1}{\Omega}\partial_T$

A strong Carroll structure also requires the addition of a connection that parallel transport both g and $\vec{v}$

These conditions do not fix the connection, making the choice ambiguous !!

Here we will use a connection that defines a parallel transport for horizontal vectors, and respect the time / space splitting.

The latter defines two distinct time and space Carroll covariant derivatives

acting on Carrollian tensors that produce

[illegible]

new Carrollian tensor.

1) $\hat{\nabla} \equiv \hat{\partial} + \gamma$ such that on scalar acts as

$$\hat{\nabla}_i \Phi = \hat{\partial}_i \Phi$$

and its action on vector is

$$\hat{\nabla}_i V^j = \hat{\partial}_i V^j + \hat{\gamma}^j_{ik} V^k$$

Metric compatibility requires

$$\hat{\gamma}^j_{[ik]} = 0$$

$$\Rightarrow \quad \boxed{\hat{\nabla}_i a_{jk} = 0}$$

Here, $\hat{\gamma}^i_{jk}$ is the Levi-Civita-Carroll connection and is defined as

$$\boxed{\gamma^i_{jk} = \frac{a^{il}}{2}\left(\hat{\partial}_j a_{kl} + \hat{\partial}_k a_{jl} - \hat{\partial}_l a_{jk}\right)}$$

2) For the Time covariant derivative we have

$$\frac{1}{\Omega}\hat{D}_T \Phi = \frac{1}{\Omega}\partial_T \Phi$$

$$\frac{1}{\Omega}\hat{D}_T V^i = \frac{1}{\Omega}\partial_T V^i + \hat{\gamma}^i{}_j V^j$$

Such that $$\frac{1}{\Omega}\hat{D}_T a_{ij} = 0$$

The temporal Carroll connection is defined as

$$\hat{\gamma}^i{}_j = \underbrace{\frac{1}{2\Omega}\partial_T a_{ij}}_{\text{Extrinsic curvature } \frac{1}{2}\mathcal{L}_v a_{ij}} = \xi_{ij} + \frac{1}{d} a_{ij}\,\theta$$

- $\xi_{ij} \equiv$ Carrollian shear

- $\theta \equiv$ Carrollian expansion

$$\theta = \hat{\gamma}^i{}_i = \frac{1}{\Omega}\partial_T \ln\sqrt{a}$$

Other intrinsic geometric quantities

This can be obtained from the commutators of the derivatives

1) $[\hat{\nabla}_i , \hat{\nabla}_j]\Phi = \frac{2}{\Omega}\omega_{ij}\partial_T\Phi$

2) $[\frac{1}{\Omega}\hat{\partial}_T , \hat{\nabla}_i]\Phi = \varphi_i \frac{1}{\Omega}\partial_T\Phi$

Torsion components

where $\varphi_i = \frac{1}{\Omega}(\partial_T b_i + \partial_i\Omega)$ acceleration

$\omega_{ij} = \partial_{[i}b_{j]} + b_{[i}\varphi_{j]}$ vorticity

3) $[\hat{\nabla}_k , \hat{\nabla}_l]V^i = \hat{R}^i{}_{jkl}V^j + 2\omega_{kl}\frac{1}{\Omega}\hat{\partial}_T V^i$

where $\hat{R}^i{}_{jkl} \equiv$ Carroll-Riemann Tensor

$\hat{R}^i{}_{ikl} = \hat{R}_{kl} \equiv$ Carroll-Ricci Tensor

$$\hat{R}^{l}{}_{jil} = \hat{R}_{jl}$$

non-symmetric

$$a^{ij}\hat{R}_{ij} = \hat{R} \equiv$$ Carroll-Ricci scalar

For $d=2$ the Carroll-Ricci tensor can be decomposed as

$$\hat{R}_{ij} = \hat{K}\, a_{ij} + \hat{A}\, \eta_{ij}$$

where

$$\hat{K} = \frac{1}{2} a^{ij}\hat{R}_{ij} \;, \quad \hat{A} = \frac{1}{2}\eta^{ij}\hat{R}_{ij}$$

with $\eta_{ij} = \sqrt{a}\,\epsilon_{ij}$

Carroll Geometry on a null hypersurface

The idea now is to show that the same Carroll structure that we discussed before can be obtained on the null hypersurface of a Lorentzian spacetime.

Consider a $d+2$ Lorentzian spacetime foliated with null hypersurfaces, equipped with the metric

$$ds^2 = g_{ab}\, dx^a dx^b$$

$$= -2\,\Box\,\Omega\left(dT - b_i dx^i + \theta^T dr - b_i \theta^i dr\right) dr$$

$$+ g_{ij}\left(dx^i + \theta^i dr\right)\left(dx^j + \theta^j dr\right)$$

where the functions Ω, $\Box$, b_i, θ^T, θ^i and g_{ij} depend on all coordinates (r, T, x^i)

- We have a $d+1$ null hypersurface at each constant r with degenerate metric $g_r = g_{ij}(r,T,x)\, dx^i dx^j$

- The form of the $d+2$ metric is preserved by the diffeomorphisms

$r \to r'(r)$ $\quad T \to T'(r,T,x)$)

Residual gauge symmetry

$$x \to x'(r, x)$$

with Jacobian

$$J^a{}_b = \frac{\partial x'^a}{\partial x^b}$$

Under these diffeomorphisms, the functions that parameterized the d+2 metric transform as

$$\Omega' = (J^\tau{}_\tau)^{-1}\Omega$$

$$b'_i = (J^{-1})^i{}_j (J^\tau{}_\tau b_i + J^\tau{}_i)$$

$$g'_{ij} = (J^{-1})^k{}_i (J^{-1})^l{}_j \, g_{kl}$$

These functions transform at every constant r null surface as they do on a Carrollian spacetime!!

(Carroll diffeos at every leaf)

$$\sqcap' = (J^r{}_r)^{-1} \sqcap$$

$$\theta'^\tau = (J^\tau{}_r)^{-1}(J^\tau{}_\tau \theta^\tau - J^\tau{}_r + J^\tau{}_i \theta^i)$$

$$\theta^{i} = (J^{r}{}_{r})^{-1}(J^{r}{}_{j}\theta^{j} - J_{r})$$

The last three functions give account for the non-trivial r-dependence of the residual gauge symmetry. We can fix the gauge to make

$$\mathcal{J} = \underline{1}, \quad \theta^{T} = 0, \quad \theta^{i} = 0$$

which is achievable by restricting the diffeomorphism to

$$r \to r, \quad T \to T'(T, x), \quad x \to x'(x)$$

The Lorentzian metric symplifies to

$$ds^2_M = -2\Omega(dT - b_i dx^i)\,dr + g_{ij}\,dx^i dx^j$$

So at each constant r we find:

- null hypersurface C_r with degenerate metric $ds^2_{C_r} = g_{ij}\,dx^i dx^j$

- Ω, b_i and g_{ij} transform as under a Carrollian diffeos !!

Now we need the Carroll vector and the Ehresmann connection.

Our Lorentzian metric allows to define two null vector fields $\in TM$

$$\boxed{\vec{l} = \frac{1}{\Omega}\partial_\tau \quad \text{and} \quad \vec{m} = \partial_r}$$

with $\vec{l}\cdot\vec{m} = 1$

Their dual forms are

$$\boxed{\underline{l} = -dr \quad \text{and} \quad \underline{m} = \Omega\left(d\tau - b_i\,dx^i\right)}$$

- $\vec{l}$ normal to C_r
- $\vec{l} \in TC_r$ (since is also tangent to C_r)

- $g(l) = 0$, Kernel of $dS_{C_r} = g_{ij}\, dx^i dx^j$

We can also make a decomposition of vector at each constant-r surface C_r in terms of a vertical and horizontal part

Here

$$X \in T C_r = V_r \oplus H_r$$

1) $\vec{l} \in V_r$, vector tangent to $\vec{l}$ belongs to V_r

$$\rightarrow \boxed{X_V = \frac{X^T}{\Omega} \partial_T = X^T E \in V_r}$$

2) Vectors $X \in H_r$ are defined in

$$\boxed{\vec{X}_H = X^i\left(\partial_i + \frac{b_i}{\Omega}\partial_T\right) = X^i E_i \in H_r}$$

and this decomposition is obtained from the condition

$$\boxed{\vec{X}_H \cdot \vec{l} = 0 \,,\; \vec{X}_H \cdot \vec{m} = 0}$$

So $b_i(r,T,x)$ plays the role of the Ehresmann Connection at each constant-r surface!!

We can also compute the extrinsic geometry for this embedding. For this we define the projector onto H_r

$$\underline{h_a{}^b = \delta^a{}_b - m^a l_b - l^a m_b}$$

The non-vanishing extrinsic quantities

$$D^{ab} = \frac{1}{2} h^{ac} h^{bd} \mathcal{L}_{\vec{\ell}}\, h_{cd} \qquad \text{(Deformation tensor)}$$

$$\omega_a = h^b{}_a n_c \nabla_b \ell^c \qquad \text{(Twist)}$$

D^{ab} can be decomposed as

$$\Theta = h_{ab} D^{ab} = \frac{1}{2} h^{ab} \mathcal{L}_{\vec{\ell}}\, h_{ab}$$

$$\sigma^{ab} = D^{ab} - \frac{\Theta}{d} h^{ab}$$

For our geometry at hand, one can show that the above quantities coincide with the Carrollian ones at every value of r

$$\omega_i = -\frac{1}{2} \varphi_i$$

$$\Theta = \theta$$

$$\sigma_{ij} = \xi_{ij}$$

$q_{ij} = \delta_{ij}$

Final message: Our definition of Carroll spacetime is adapted to the description of families of null hypersurfaces embedded on a Lorentzian spacetime

(Conformal) Carroll isometries

Isometries are diffeomorphism that leave the spacetime invariant. In general they satisfy

$$\mathcal{L}_\xi g_{\mu\nu} = 0 \,, \quad \xi \text{ Killing vector}$$

For a Carrollian structure the latter translates to

$$\boxed{\mathcal{L}_\xi \vec{v} = 0 \,, \quad \mathcal{L}_\xi a_{ij} = 0}$$

with $\xi = \xi^T(T,x)\,\partial_T + \xi^i(x)\,\partial_i$ absolute space

$$= \xi^{\hat{T}}\,\frac{1}{\Omega}\,\partial_T + \xi^i\,\hat{\partial}_i \,, \quad \text{with } \xi^{\hat{T}} = \xi^T - \xi^i\,\frac{b_i}{\Omega}$$

The latter gives rise to two Carroll-Killing equations

$$\boxed{\begin{aligned} &\frac{1}{\Omega}\,\partial_T\,\xi^{\hat{T}} + \varphi_i\,\xi^i = 0 \\ &\hat{\nabla}_{(i}\,\xi^k a_{j)k} + \xi^{\hat{T}}\,\hat{\gamma}_{ij} = 0 \end{aligned}}$$

Example: Flat Carroll spacetime

$\Omega = 1$, $a_{ij} = \delta_{ij}$, $T = -dT + b_i dx$ $\hookrightarrow$ cte

$$ds^2 = \delta_{ij} dx^i dx^j \quad , \quad v = \partial_T$$

The Carroll-Killing reduced to

$$\partial_T \xi^T = 0 \quad \longrightarrow \quad \xi^T = f(x)$$

$$\partial_i \xi_j + \partial_j \xi_i = 0 \longrightarrow \text{linear in } x^i$$

$$\Rightarrow \quad \xi = f(x)\partial_T + (\Omega_j{}^i x^j + X^i)\partial_i$$

infinite number of solutions for a weak Carroll structure

For a strong Carroll structure we also have the condition that the connection is invariant under Carroll isometries. For a flat connection this implies $\delta_\xi \hat{\gamma}^k_{ij} = 0$, $\delta_\xi \hat{\gamma}_{ij} = 0$

$$\partial_i \partial_j f = 0$$ $\to$ f(x) is at most linear in x

$$\to \quad f(x) = T + B_i x^i$$

Carroll boost

$$\xi = (T + B_i x^i)\partial_T + (\Omega_j{}^i x^j + X^i)\partial_i$$

Time translation — rotation — space-translation

For a conformal isometry the conditions are

$$\boxed{\mathcal{L}_\xi v = \mu v \,, \quad \mathcal{L}_\xi a_{ij} = \lambda a_{ij}}$$

where $\mu(T,x)$ and $\lambda(T,x)$ functions of (T,x)

So

$$\boxed{\mu = -\left(\frac{1}{\Omega}\partial_T \xi^T + \varphi_i \xi^i\right)}$$

The second Killing equation yields

$$\boxed{\hat{\nabla}_{(i}\xi^k a_{j)k} + \xi^{\hat{t}}\hat{\gamma}_{ij} = \lambda a_{ij}}$$

and from the trace of $\mathcal{L}_\xi a_{ij} = \lambda a_{ij}$ we get

$$\boxed{\lambda = \frac{2}{d}\left(\hat{\nabla}_i \xi^i + \theta \xi^{\hat{t}}\right)}$$

All the latter comes with the extra condition

$$\boxed{2\mu + \lambda = 0} \longrightarrow$$

Comes from Weyl covariance

The conformal Killing should be insensitive to rescalings of the metric

<u>Simplest case</u>: $\Omega = 1$, $b_i = cte$, $d = 2$

$$a_{ij}dx^i dx^j = q_{ij}dx^i dx^j$$

$$= d\theta^2 + \sin^2\theta\, d\varphi^2$$

$$\boxed{\xi = \left(T(x) + \frac{\tau}{2}\partial_i Y^i\right)\partial_\tau + Y^i(x)\partial_i}$$

Infinite dimensional

$Y^i(x) \equiv$ Conformal Killings of the sphere

$T(x) \equiv$ supertranslations

These generators form the Conformal Carroll algebra

$$\underline{\underline{CCarr(3) \equiv T \ltimes SO(3,1)}}$$

The latter is isomorphic to

$$BMS_4$$

One can compute the asymptotic Killing vectors of Ricci-flat spacetime and take the $r \to \infty$ limit. Then one finds again

$$\xi = \left(T + \frac{T}{2}\partial_i \Psi^i\right)\partial_T + \Psi^i \partial_i$$

Consider an action

$$S = \int_M d^{d+1}x \sqrt{-g}\, \mathcal{L}(g_{\mu\nu}, \underbrace{\Phi}_{\text{other fields}})$$

defined on a pseudo-Riemannian manifold M with metric $g_{\mu\nu}$ (Lorentzian signature).

The variation of the action yields

$$\delta S = \int_M d^{d+1}x \sqrt{-g} \left(EOM\, \delta\Phi + \frac{1}{2} T^{\mu\nu} \delta g_{\mu\nu} \right)$$

$$+ B.T.$$

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}}$$

on-shell we have

$$EOM = 0$$

If the action is invariant under diffeomorphisms generated by

$$\xi = \xi^{\mu}(T,x)\,\partial_\mu$$

that transform the geometry as

$$\delta_\xi g_{\mu\nu} = -\mathcal{L}_\xi g_{\mu\nu}$$

$$= -(\partial_\mu \xi_\nu + \partial_\nu \xi_\mu),$$

then from the on-shell variation of the action we get

$$\delta_\xi S = \frac{1}{2}\int_M d^{d+1}x \sqrt{-g}\, T^{\mu\nu}\delta_\xi g_{\mu\nu}$$

$$= \int_M d^{d+1}x \sqrt{-g}\, \xi_\nu \nabla_\mu T^{\mu\nu}$$

$$- \int_M d^{d+1}x\, \partial_\mu (T^{\mu\nu}\xi_\nu).$$

Diffeo invariance implies the conservation of the energy-momentum tensor

$$\boxed{\nabla_\mu T^{\mu\nu} = 0.}$$

Additionally, if our action is also Weyl invariant, where a Weyl transformation on the metric reads

$$g_{\mu\nu} \longrightarrow \frac{1}{B^2} g_{\mu\nu}$$

with $B = B(T, x)$, then we find

$$\delta_B S = - \int_M d^{d+1}x \sqrt{-g} \ln B \, T^\mu{}_\mu$$

$$\boxed{\text{Weyl invariance} \longleftrightarrow T^\mu{}_\mu = 0}$$

How does the latter translate to Carroll ??

We start with the action

$\Omega\sqrt{a}\,d\tau\, d^dx$ in the volume in C

$$S = \int_C d\tau\, d^dx \sqrt{a}\,\Omega\,\mathcal{L}(\Omega, b_i, a_{ij})$$

that is invariant under Carrollian diffeos generated by

$$\xi = \xi^\tau(\tau,x)\partial_\tau + \xi^i(x)\partial_i$$
$$= \xi^{\hat\tau}\frac{1}{\Omega}\partial_\tau + \xi^i\hat\partial_i$$

$$\xi^{\hat\tau} = \xi^\tau - \xi^i\frac{b_i}{\Omega}$$

The variation of the action with respect to the building blocks of the Carroll structure gives a set of Carroll momenta

$$\Pi^{ij} = \frac{2}{\Omega\sqrt{a}}\frac{\delta S}{\delta a_{ij}} \quad \text{(Carroll Energy-stress Tensor)}$$

$$\Pi^i = \frac{1}{\Omega\sqrt{a}}\frac{\delta S}{\delta b_i} \quad \text{(Carroll Energy-flux)}$$

$$\Pi = -\frac{1}{\sqrt{a}}\left(\frac{\delta S}{\delta\Omega} + \frac{b_i}{\Omega}\frac{\delta S}{\delta b_i}\right) \quad \text{(Carroll Energy density)}$$

The latter is analogous to ... T

The latter is analogous to a stress-tensor.

Hence the on-shell variation of the action is

$$\boxed{\delta S = \int_M d\tau\, d^d x \sqrt{a}\,\Omega \left(\frac{1}{2} \pi^{ij} \delta a_{ij} + \pi^i \delta b_i - \frac{1}{\Omega} \left(\pi + b_i \pi^i \right) \delta \Omega \right)}$$

Under Carrollian diffeos, the geometry transforms as

$$\delta_\xi a_{ij} = -\mathcal{L}_\xi a_{ij} = -\left(2 \hat{\nabla}_{(i} \xi^k a_{j)k} + 2 \xi^\tau \hat{\gamma}_{ij} \right)$$

$$\delta_\xi v = -\mathcal{L}_\xi v = \left(\frac{1}{\Omega} \partial_\tau \xi^\tau + \varphi_i \xi^i \right) v$$

$$\delta_\xi T = -\mathcal{L}_\xi T = \left(\frac{1}{\Omega} \partial_\tau \xi^\tau + \varphi_i \xi^i \right) T + \left(\hat{\partial}_i \xi^\tau - \varphi_i \xi^\tau - 2 \xi^j \omega_{ji} \right) dx^i$$

From the latter we can find the transformation for Ω and b_i

$$-\frac{1}{\Omega} \delta_\xi \Omega = \frac{1}{\Omega} \partial_\tau \xi^\tau + \varphi_i \xi^i$$

$$\boxed{-\delta_\xi b_i = b_i\left(\frac{1}{\Omega}\partial_T \xi^{\hat{T}} + \varphi_j \xi^j\right) - (\hat{\partial}_i - \varphi_i)\xi^{\hat{T}} + 2\xi^j \omega_{ji}}$$

Then, the diffeomorphic variation of the action gives

$$\delta_\xi S = \int_M dT\, d^dx \sqrt{a}\,\Omega \left(\Pi^{ij} \delta_\xi a_{ij} + \Pi^i \delta_\xi b_i - \frac{1}{\Omega}(\Pi + b_i \Pi^i)\delta_\xi \Omega \right)$$

$$= \int_M dT\, d^dx \sqrt{a}\,\Omega \Big[-\xi^{\hat{T}}\Big(\left(\frac{1}{\Omega}\partial_T + \theta\right)\Pi + (\hat{\nabla}_i + 2\varphi_i)\Pi^i$$

$$+ \Pi^{ij}\hat{\gamma}_{ij} \Big) + \xi^i\Big((\hat{\nabla}_j + \varphi_j)\Pi^j{}_i + 2\Pi^j \omega_{ji} + \Pi \varphi_i \Big)\Big]$$

$$+ \text{B.T.}$$

$\delta_\xi S = 0$ implies

$$\boxed{\left(\frac{1}{\Omega}\partial_T + \theta\right)\Pi + (\hat{\nabla}_i + 2\varphi_i)\Pi^i + \Pi^{ij}\hat{\gamma}_{ij} = 0}$$

"Equation for the energy density"

$$\boxed{\left(\hat{\nabla}_j+\varphi_j\right)\Pi^j{}_i+2\Pi^j\omega_{ji}+\Pi\varphi_i=-\left(\frac{1}{\Omega}\partial_T+\theta\right)P_i}$$

"Equation for the evolution of the momentum P_i"

P_i is the momentum and it appears in the variation of the action as a boundary term

$$\partial_T\left(\sqrt{a}\,\xi^i P_i\right)=\sqrt{a}\,\Omega\,\xi^i\left(\frac{1}{\Omega}\partial_T+\theta\right)P_i$$

due to the time independence of ξ^i.

Weyl invariance

Weyl transformation act on the geometric data as

$$a_{ij}\to\frac{1}{B^2}a_{ij}\,,\quad \Omega\to\frac{1}{B}\Omega$$

$$b_i\to\frac{1}{B}b_i$$

when $B=B(T,x)$ is an arbitrary function.

If the action is also Weyl invariant, we have that

$$\boxed{\Pi^{ij} \to \frac{1}{B^{d-1}} \Pi^{ij}, \quad \Pi^{i} \to \frac{1}{B^{d}} \Pi^{i} \qquad \Pi \to \frac{1}{B^{d+1}} \Pi, \quad P_i \to \frac{1}{B^{d}} P_i}$$

Weyl invariance of the action implies $\delta_B S = 0$

$$\Rightarrow \boxed{\Pi^i{}_i = \Pi}$$

analog to

$T^\mu{}_\mu = 0$!!

Weyl covariance also allows for a Weyl covariant derivative given a Weyl connection

In general this is defined as

$$D_\mu \equiv \nabla_\mu + A_\mu$$

where $A_\mu \to A_\mu - \partial_\mu \ln B$.

For our Carrollian structure we can define two Weyl-Carroll covariant derivatives as

$$\boxed{\frac{1}{\Omega}\mathcal{D}_T V^i = \frac{1}{\Omega}\hat{\mathcal{D}}_T V^i + \frac{w-1}{d}\theta V^i}$$

$$\boxed{\hat{\mathcal{D}}_i V^j = \hat{\nabla}_i V^j + (w-1)\varphi_i V^j + \varphi^j V_i - \delta_i^j V^l \varphi_l}$$

with

$$\theta \to B\theta - \frac{d}{\Omega}\partial_T B, \quad \varphi_i \to \varphi_i - \hat{\partial}_i \ln B$$

For a Weyl-Carroll covariant theory, the conservation laws are recast as

$$\frac{1}{\Omega}\hat{\mathcal{D}}_T \Pi + \hat{\mathcal{D}}_i \Pi^i + \Pi^{ij}\xi_{ij} = 0$$

$$\hat{\mathcal{D}}_i \Pi^i{}_j + 2\Pi^i \omega_{ij} + \left(\frac{1}{\Omega}\hat{\mathcal{D}}_T + \xi_{ij}\right)P^i = 0$$

From a limit ($c \to 0$)

The latter can also be derived from a $c \to 0$ limit of a relativistic theory with action

$$S = \int_M d^{d+1}x \sqrt{-g}\, \mathcal{L}(\Phi, g_{\mu\nu})$$

and a conserved energy momentum tensor

$$T^{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}}$$

$$\nabla_\mu T^{\mu\nu} = 0$$

For this derivation we will parameterized the metric $g_{\mu\nu}$ and the EM tensor $T^{\mu\nu}$ in the following way:

1) R-P gauge

$$\boxed{g_{\mu\nu} dx^\mu dx^\nu = -c^2(\Omega dt - b_i dx^i)^2 + a_{ij}\, dx^i dx^j}$$

2) decomposition of $T^{\mu\nu}$ with respect to a congruence

u

u_μ

$$\underbrace{T^{\mu\nu} = (\mathcal{E}+P)\frac{u^\mu u^\nu}{c^2} + P g^{\mu\nu} + \tau^{\mu\nu} + \frac{2u^{(\mu}q^{\nu)}}{c^2}}_{\text{" abstract fluid}}$$

Here u^μ is timelike such that $u_\mu u^\mu = -c^2$

we chose $\boxed{u^\mu = \frac{1}{\Omega}\partial_\tau}$ at rest !!

This choice makes simple the decomposition of $T^{\mu\nu}$ as

$$\boxed{T_{00} = \Omega^2 \mathcal{E}, \quad T_0{}^i = -\frac{\Omega}{c} q^i, \quad T^{ij} = P a^{ij} + \tau^{ij}}$$

The idea now is to make a power expansion on the conservation of $T^{\mu\nu}$.

$$\boxed{\frac{c}{\Omega}\nabla_\mu T^\mu{}_0 = E + O(c^2)}$$

$$\boxed{\nabla_\mu T^{\mu i} = \frac{1}{c^2} H^i + G^i + O(c^2)}$$

The expansion of all the geometric quantities is straightforward but for the case of $T^{\mu\nu}$ it depends on the model.

Here we assume the following behaviour

$$\mathcal{E} = \Pi + O(c^2) \qquad T_{00} = \Omega^2 \Pi + O(c^2)$$

$$q^i = \Pi^i + c^2 p^i + O(c^4) \implies T_0{}^i = -\frac{\Omega}{c}\Pi^i - c\Omega P^i + O(c^3)$$

$$p a^{ij} + \tau^{ij} = \Pi^{ij} + O(c^2) \qquad T^{ij} = \Pi^{ij} + O(c^2)$$

In this way we find

$$E = -\left(\frac{1}{\Omega}\hat{D}_T + \theta\right)\Pi - (\hat{\nabla}_i + 2\varphi_i)\Pi^i - \Pi^{ij}\hat{\gamma}_{ij}$$

$$G_i = (\hat{\nabla}_i + \varphi_i)\Pi^i{}_j + 2\Pi^i\omega_{ij} + \Pi\varphi_j + \left(\frac{1}{\Omega}\partial_T + \theta\right)P_i$$

$$H_i = \left(\frac{1}{\Omega}\hat{D}_T + \theta\right)\Pi_i + \Pi^j\hat{\gamma}_{ji}$$

*) We recover equation E and G_i just as we did from the symmetry argument.

**) In the limiting procedure, P^i appears explicitly as the subleading term of $T_0{}^i$.

***) Equation H_i was absent when we derived the conservation equations from the symmetry analysis, similar to what happened with P_i.

From the relativistic ascendant, the latter is a remnant of the full diffeo invariance.

Weyl invariance

$$T^\mu{}_\mu = 0 \implies \Pi = \Pi^i{}_i \quad \text{at leading order}$$

and $\mathcal{E} = 2P$

The conservation equations are now

$$\boxed{\begin{aligned} &\frac{1}{\Omega}\hat{D}_T \Pi + \hat{D}_i \Pi^i + \Pi^{ij}\xi_{ij} = 0 \\ &\hat{D}_i \Pi^i{}_j + 2\Pi^i \omega_{ij} + \left(\frac{1}{\Omega}\hat{D}_T + \xi_{ij}\right)P^i = 0 \\ &\frac{1}{\Omega}\hat{D}_T \Pi_j + \Pi_i \xi^i{}_j = 0 \end{aligned}}$$

More degrees of freedom?

We could have additional degrees of freedom by making different assumptions in the behaviour of $T^{\mu\nu}$ with respect to c. We will see that for a Carrollian fluid which is the holographic dual of a Ricci-flat spacetime, there is an additional stress tensor Σ_{ij} that appears at $\frac{1}{c^2}$ order of T^{ij}

Isometries and (non-) conservation laws

Noether

Symmetry $\Longrightarrow$ Conserved current J^μ

$\Downarrow$

conserved charge Q

Given an isometry generated by ξ^μ such that

$$\mathcal{L}_\xi g_{\mu\nu} = 0$$

one can construct the following current

$$\boxed{I^\mu = T^{\mu\nu}\xi_\nu}$$

which is conserved

$$\boxed{\nabla_\mu I^\mu = 0}$$

Integration over a spacelike hypersurface Σ_d embedded in M provides a definition of a conserved charge as

$$\boxed{Q[\xi] = \int_{\Sigma_d} d^d x \sqrt{\sigma}\, n_\mu J^\mu}$$

$\sigma_{\mu\nu} \to$ metric of Σ_d

$n_\mu \to$ normal to Σ_d

Such that $\frac{d}{dT} Q = 0$ //

This construction is also valid for the presence of conformal isometries.

How does the latter work for Carrollian isometries ??

For a Carroll isometry whose generators satisfy

$$\boxed{\mathcal{L}_\xi \vec{v} = 0 \quad , \quad \mathcal{L}_\xi a_{ij} = 0}$$

Current $\equiv$ $\kappa = ?$, $K^i = ?$

$\mathrm{Div}(\kappa, K^i)$?? is there conservation ??

One way to derive the latter is through the expansion of the relativistic counterpart in power of c.

Given
$$T_{00} = \Omega^2 \Pi + \mathcal{O}(c^2)$$
$$T_0{}^i = -\frac{\Omega}{c}\Pi^i - c\Omega P^i + \mathcal{O}(c^3)$$
$$T^{ij} = \Pi^{ij} + \mathcal{O}(c^2)$$

and $g_{\mu\nu}$ in the R-P gauge, the current

$$\boxed{\begin{aligned} I_0 &= -c\Omega\kappa + \mathcal{O}(c^3) \\ I^i &= K^i + \mathcal{O}(c^2) \end{aligned}}$$

Remember!!

$$\xi^\mu \partial_\mu = \xi^T(T,x)\partial_T + \xi^i(T,x)\partial_i$$

and

$$\xi^i(T,x) = \xi^i(x) + c^2 V^i(T,x) + \mathcal{O}(c^4)$$

with

$$\boxed{\begin{aligned} K &= \xi^i P_i - \xi^{\hat T}\Pi \\ K^i &= \xi^j \Pi_j{}^i - \xi^{\hat T}\Pi^i \end{aligned}}$$

Components of a Carroll current associated to an isometry

From the divergence of $\nabla_\mu I^\mu$ follows

$$\nabla_\mu I^\mu = \mathcal{K} + O(c^2)$$

with

$$\boxed{\mathcal{K} = \left(\frac{1}{\Omega}\partial_T + \theta\right)K + \left(\hat\nabla_i + \varphi_i\right)K^i}$$

The latter is the Carrollian analog of the divergence of a current. One would expect that $\mathcal{K}=0$ if $\xi = \xi^{\hat T}\partial_T + \xi^i\hat\partial_i$ generates an isometry. Using the conservation equations we find that

$$\mathcal{K} = -\Pi^i\left((\partial_i - \varphi_i)\xi^T - 2\xi^j\omega_{ji}\right)$$

$$\boxed{\mathcal{K} = -\Pi^i \delta_\xi T}$$

The conservation of the current given by $\mathcal{K}$ and K^i happens only when

1) $\delta_\xi T = 0$ (strong Carrollian Killing vector)

2) $\Pi^i = 0$ (no energy flux)

We can also define a charge Q associated to $\mathcal{K}$ and K^i as the following integration

$$\boxed{Q_K = \int_{\Sigma_d} d^d x \sqrt{a}\,(\mathcal{K} + b_i K^i)}$$

that obeys

boundary term

$$\frac{dQ_K}{dT} = \int_{\Sigma_d} d^d x \sqrt{\sigma}\, \Omega\, \mathcal{K} - \int_{\partial\Sigma} * K\, \Omega$$

Q_K is conserved only if $\mathcal{K} = 0$!!

Carrollian fluids at the event horizon

The membrane paradigm relates the black hole event horizon with a membrane that lives and evolves in a three dimensional spacetime.
The latter is formulated by constructing a 2+1 dimensional timelike surface near the horizon called "stretched horizon".

In the original derivation, the authors relate the dynamical equations of the membrane in the near-horizon limit with the one of a Galilean fluid.

Here we show that the near horizon limit is better interpreted as an ultrarelativistic limit of the stretched horizon, where its dynamics is described by Carrollian fluid equations.

This is based on the paper: 1903.0965[illegible] paper: 1903.0965 (L. Donnay and C. Marteau)

Near horizon geometry

Here we choose null Gaussian coordinates

$$\boxed{\begin{aligned} ds^2 &= -2\kappa\rho\, dv^2 + 2\, d\rho\, dv + 2\,\theta_A \rho\, dv\, dx^A \\ &\quad + (G_{AB} + \lambda_{AB}\rho)\, dx^A dx^B + \mathcal{O}(\rho^2) \end{aligned}}$$

where κ, θ_A, G_{AB} and λ_{AB} depend on all coordinates.

- ρ is the radial coordinate
- v is the advanced time
- Surfaces of constant v an ρ are $(D-2)$-dimensional spheres parameterized by x^A.
- $v = $ cte defines null hypersurfaces

  $\rho = $ cte defines timelike hypersurfaces.

- The horizon is located at $\rho = 0$

- G_{AB} is used to raise and lower spatial indices

One can easily see that at $\rho \to 0$, the induced metric on the horizon $\mathcal{H}$ is

$$\boxed{ds^2_{\mathcal{H}} = 0 \cdot dv^2 + 0 \cdot dv\, dx^A + G_{AB}\, dx^A\, dx^B.}$$

This can be interpreted as the degenerate metric of the Carroll structure.

What about the rest of the pieces that are part of the Carroll structure?

We can decompose the bulk metric in the following way:

$$\boxed{g_{ab} = q_{ab} + L_a N_b + N_a L_b}$$

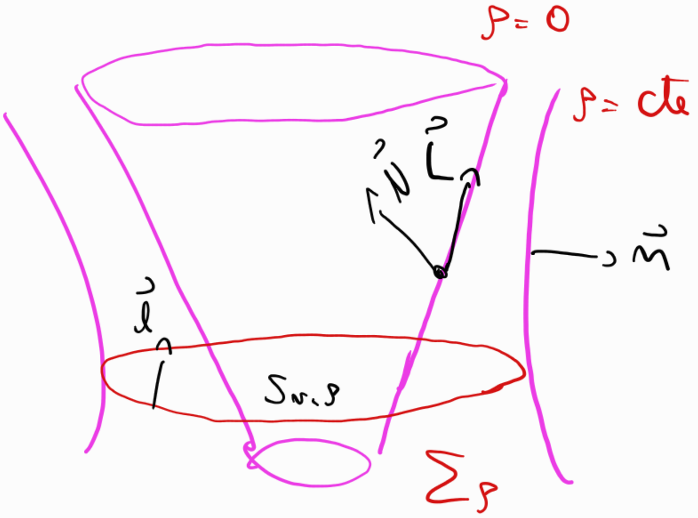
adapted to the null hypersurfaces

- $\vec{L} = L^a \partial_a = \partial_v - \rho\,\theta^A \partial_A + \kappa\rho\,\partial_\rho$

  is a null vector normal to the horizon

- $N = N_a dx^a = dv$

  $N^a \partial_a$ is a null vector transverse to the horizon



$\vec{m}$ spacelike normal to Σ_ρ

$\vec{\ell}$ timelike vector normal to v section of Σ_ρ

- $\boxed{N(\vec{L}) = 1}$ $\quad$ $\vec{L}$ and $\vec{N}$ allow to define the extrinsic curvature elements of $\mathcal{H}$.

- q_{ab} is the projector perpendicular to $\vec{L}$ and $\vec{N}$

These two vectors complete the Carroll structure on the horizon.

$$\boxed{\begin{aligned}
&\mathcal{M} \xrightarrow{\rho=0} \mathcal{H} \\
&g = ds^2_{\mathcal{H}} = 0\cdot dv^2 + 0\cdot dv\,dx^A + G_{AB}\,dx^A dx^B \\
&\vec{v} = \vec{L}_{\mathcal{H}} = \partial_v \\
&\tau = N_{\mathcal{H}} = dv
\end{aligned}}$$

In the language we have been using, the latter corresponds to

$$\boxed{a_{AB} = G_{AB}\,, \quad b_A = 0\,, \quad \Omega = 1}$$

The extrinsic geometry of the horizon is captured by the triplet $(\Sigma_{AB}, \omega_A, \bar{\kappa})$ with

a) Σ … Deformation tensor

1) $\Sigma_{AB} = \frac{1}{2} q^a{}_A \, q^b{}_B \, \mathcal{L}_{\vec{L}} \, q_{ab}$ (second fundamental form)

2) $\omega_A = q^a{}_A \left(N_b \nabla_a L^b \right)$ Twist field (Hajicek one-form)

3) $L^b \nabla_b L^a = \hat{\kappa} \, L^a$ Surface gravity

Using our bulk metric we find

$$\underbrace{\Sigma_{AB} = \frac{1}{2} \partial_r G_{AB}}_{\hat{\gamma}_{AB}} \, , \quad \omega_A = -\frac{1}{2} \theta_A$$

$$\underline{\hat{\kappa} = \kappa}$$

From the decomposition of Σ_{AB} we find

- $\Theta = G^{AB} \Sigma_{AB} = \partial_r \ln \sqrt{G}$ Expansion

measures the rate of variation of the surface element of the spatial section of $\mathcal{H}$

- $\sigma_{AB} = \frac{1}{2}\partial_r G_{AB} - \frac{\Theta}{D-2} G_{AB}$ Shear (Deformation)

Θ is positive everywhere on $\mathcal{H}$

$\Rightarrow$ Area only increases

Dynamical equations on the horizon

The projection of the vacuum Einstein's equations give rise to two equations:

1) $L^a L^b R_{ab} = 0$ null Raychaudhuri equation

2) $X^a{}_A L^b R_{ab} = 0$ Damour equation

Using our near horizon geometry, we get at

$\rho = 0$

$$\boxed{\partial_v \Theta - \kappa \Theta + \frac{\Theta^2}{D-2} + \sigma_{AB}\sigma^{AB} = 0}$$

which describes the evolution of the expansion along the null geodesic congruence $\vec{L}$.

For the Damour equation we get

$$\boxed{(\partial_v + \Theta)\theta_A + 2\nabla_A\left(\kappa + \frac{D-3}{D-2}\Theta\right) - 2\nabla_B\sigma^B{}_A = 0}$$

with $\nabla_B \equiv \nabla_B[G_{AB}]$

In the membrane paradigm, the above equation has been interpreted as a $(D-2)$-dimensional

Navier-Stokes equation for a viscous fluid.

There is no spatial velocity v^A since our fluid is at rest (Co-moving frame $\underset{\sim}{L}_{\mathcal{H}} = \partial_v$)

We will see that these two equations are Carrollian conservation equations that are obtained as an ultrarelativistic limit.

Stretched horizon and its Carrollian limit

A stretched horizon consist on a codimension-one timelike hypersurface Σ_ρ of constant ρ (very small).

This surface becomes null when $\rho = 0$

$\Rightarrow$

For $\rho > 0 \longrightarrow$ relativistic spacetime

For $\rho = 0 \longrightarrow$ Carrollian spacetime

$$\boxed{\rho \to 0 \;=\; c \to 0}$$

For a timelike hypersurface Σ_ρ near $\rho = 0$,

its normal is given by

$$\boxed{n = \frac{d\rho}{\sqrt{2\kappa\rho}} + \mathcal{O}(\rho)}$$

we can define the extrinsic curvature as

$$\boxed{K^a{}_b = h^c{}_b \nabla_c n^a \,, \quad K = K^a{}_a}$$

where $\underline{\underline{h_{ab} = g_{ab} - n_a n_b}}$

and the stress tensor conjugate to the

induced metric

$$\underbrace{T_{ab} = -\frac{1}{8\pi G}\left(K_{ab} - h_{ab}K \right)}_{\text{"Membrane energy-momentum tensor"}}$$

Einstein's equations ensure its conservation

$$\bar{\nabla}_\mu T^{\mu\nu} = 0 \,, \qquad x^\mu = \{ v, \vec{x} \}$$

$$\bar{\nabla} \equiv \bar{\nabla}[h]$$

The membrane is then interpreted as a relativistic fluid defined on a $(D-1)$ spacetime given by Σ_ρ.

As we mentioned, going to the horizon is equivalent to take the $c \to 0$ limit. Indeed, identifying

$\rho = \tilde{c}^2$, namely, it plays the role of a "speed of light" we can expand $T^{\mu\nu}$ and $\bar{\nabla}_\mu T^{\mu\nu}$ in powers of ρ to find the corresponding Carrollian conservation equations.

In our configuration we find

$$\boxed{\begin{aligned} T_{vv} &= \Pi + \mathcal{O}(\rho) \\ T_v{}^A &= -\sqrt{\rho}\, P^A + \mathcal{O}(\rho) \\ T^{AB} &= \frac{1}{\sqrt{\rho}} \Pi^{AB} + \mathcal{O}(\rho) \end{aligned}}$$

where

$$\boxed{\Pi = -\frac{\sqrt{2\kappa}}{16\pi G}\, \Theta} \quad , \quad \boxed{\Pi^A = 0} \Bigg\}$$ Carroll Boost invariance!!

$$\boxed{p^A = \frac{1}{8\pi G\sqrt{2\kappa}}\left(\partial_A \kappa + \theta^B \Sigma_{BA} + \frac{\theta_A \partial_r \kappa}{2\kappa}\right)}$$

$$\Pi^{AB} = P G^{AB} - \Xi^{AB} \rightarrow \text{dissipative part}$$

with

$$\boxed{P = \frac{1}{8\pi G\sqrt{2\kappa}}\left(\kappa - \frac{1}{2\kappa}\partial_r \kappa + \frac{D-3}{D-2}\Theta\right)}$$

$$\boxed{\Xi_{AB} = -\frac{1}{16\pi G\sqrt{2\kappa}}\sigma_{AB}}$$

1) The energy density is proportional to the expansion of the horizon

2) The pressure is related to the gravitational pressure

$$\boxed{\mu = \kappa + \frac{D-3}{D-2}\Theta}$$

3) The dissipative part of the energy-stress

tensor is proportional to the area of the horizon

4) The current receives contribution from the surface gravity, which is interpreted as a local temperature of the horizon, and the twist

They satisfy Carrollian conservation equations:

$$\left(\partial_v+\theta\right)\Pi+\Pi^{AB}\underbrace{\Sigma_{AB}}_{\hat{\gamma}_{AB}}=0$$

$$\left(\partial_v+\theta\right)P_A-\nabla_B\Pi^B{}_A=0$$

covariant with respect to Carrollian diffeos

Also Carroll boost covariant!!

which come from the leading contribution of

$$\bar{\nabla}_\mu T^\mu{}_\nu=0.$$

Spacetime reconstruction from the boundary

The case of AAdS$_{d+1}$ spacetimes

Here we consider General relativity with negative cosmological constant Λ. Vacuum solutions must satisfy

$$\boxed{\mathcal{E}_{AB} = R_{AB} - \frac{1}{2} R g_{AB} + \Lambda g_{AB} = 0} \rightarrow$$

$$\Rightarrow R_{AB} = \frac{2}{d-1} \Lambda g_{AB}$$

with

$$\boxed{\Lambda = - \frac{d(d-1)}{2 l^2}}$$

- AdS_{d+1} is the maximally symmetric solution to Einstein equations

$$\boxed{R^{AB}{}_{CD} = -\frac{1}{\ell^2}\,\delta^{[AB]}_{[CD]}}$$

- An asymptotically AdS spacetime are the ones that in the asymptotic region become AdS

$$\text{AdS/CFT} \equiv \text{AAdS}_{d+1} \longleftrightarrow \underbrace{\text{CFT}_d}_{\substack{\text{defined at}\\ \text{the conformal boundary}}}$$

An AAdS spacetime is fully reconstructed in term of the boundary data

This is better appreciated in the FG gauge

$$\boxed{ds^2 = \frac{\ell^2}{\rho^2} d\rho^2 + \sum_{s \geq -2} \frac{\ell^s}{\rho^s} G^{(s)}_{\mu\nu}(x)\, dx^\mu dx^\nu}$$

*) $G^{(-2)}_{\mu\nu} = g_{\mu\nu}$ Boundary metric

*) $T_{\mu\nu} = \dfrac{3}{16 G \ell} G^{(1)}_{\mu\nu}$ Energy momentum tensor of the boundary theory

*) $G^{(s)}_{\mu\nu}$ are determined at every order in the expansion in term of $g_{\mu\nu}$ and $T_{\mu\nu}$

*) Einstein equations ensure

$$\nabla_\mu T^{\mu\nu} = 0$$

[illegible]

Flat limit ??

One could be tempted to naively take the $l \to \infty$ limit in the FG gauge to obtain a Ricci-flat spacetime covariant with respect to the boundary

FG gauge does not allow an $l \to \infty$ limit

One could choose a gauge like

$$ds^2 = \frac{V}{r} du^2 - 2 du\, dr + G_{ij}(dx^i - U^i du)(dx^j - U^j du)$$

Newman Unti gauge

but the latter breaks boundary covariance

(Diffeos and Weyl rescaling)

Covariant Newman-Unti gauge (D = 4)

A better choice is to consider a relaxation of the above gauge. This corresponds to take on gauge conditions

$$\boxed{g_{rr} = 0 \quad \text{and} \quad g_{r\mu} = \frac{u_\mu}{k^2}}$$

Additionally we want

$$\boxed{g_{\mu\nu} = r^2 g_{\mu\nu} + \mathcal{O}(r)}$$

$k^2 = \frac{1}{\ell^2}$ and $\Lambda = 3k^2$

- u_μ is a timelike congruence

$$\boxed{u_\mu u^\mu = -k^2} \quad \text{and} \quad u_\mu(x)$$ boundary coordinate dependence

the latter

- In the fluid/gravity correspondence, the vector is interpreted as the fluid velocity field

- u_μ also allows to address boundary Weyl covariance.

Weyl rescaling in the boundary requires

$$\Rightarrow \quad g_{\mu\nu} \to \frac{1}{B^2} g_{\mu\nu} \qquad (w = -2)$$

$$u_\mu = \frac{1}{B} u_\mu \qquad (w = -1)$$

which should be absorbe by a redefinition of the radial coordinate

$$r \to B r$$

This requires the following modification into the NU line element

$$\boxed{-du\,dr \longrightarrow \underline{u}\,(dr + rA)}$$

$$\boxed{\quad K^2 \quad}$$

with $A \longrightarrow A - d \ln B$

a Weyl connection, which allows to define a Weyl covariant derivative on

$$\boxed{\mathcal{D} \equiv \nabla + wA}$$

metric compatible

Solving Einstein's equations

The idea is to solve $\mathcal{E}_{AB} = 0$ at each order in the radial coordinate. In this procedure the line element reads

$$ds^2 = \frac{2 u_\mu dx^\mu}{K^2}\left(dr + r A_\nu dx^\nu\right) + r^2 g_{\mu\nu} dx^\mu dx^\nu$$

$$+ r C_{\mu\nu} dx^\mu dx^\nu + \frac{1}{K^4} S_{\mu\nu} dx^\mu dx^\nu$$

$$+\sum_{s=1} \left(f_{(s)} \frac{u_\mu u_\nu}{K^4} + \frac{2u_\mu}{K^2} f_{(s)\mu} dx^\mu + f_{(s),\mu\nu} dx^\mu dx^\nu \right)$$

Order r

$$A_\mu = \frac{1}{K^2}\left(a_\mu - \frac{\Theta}{2} u_\mu\right)$$

$$K^2 C_{\mu\nu} = -2\sigma_{\mu\nu} \rightarrow$$ Shear of the congruence u_μ

$W = -1$

Order 1

$$S_{\mu\nu} = 2u_{(\mu} D_\lambda (\sigma_{\nu)}{}^\lambda + \omega_{\nu)}{}^\lambda) - \frac{R}{2} u_\mu u_\nu$$

$$+ (\sigma_{\mu\lambda} + \omega_{\mu\lambda})(\sigma_\nu{}^\lambda + \omega_\nu{}^\lambda)$$

$W = 0$

Order $\frac{1}{r}$

Here is where the information of $T^{\mu\nu}$ appears:

indeed, taking the decomposition

$$\boxed{T_{\mu\nu} = \frac{3}{2}\varepsilon \frac{u_\nu u_\mu}{K^2} + \frac{\varepsilon}{2} g_{\mu\nu} + \tau_{\mu\nu} + \frac{2}{K^2} u_{(\mu} q_{\nu)}}$$

we find that

$$\boxed{f_{(1)} = 8\pi G \varepsilon}$$

$$\boxed{f_{(1)\mu} = \frac{16\pi G}{3K^2}\left(q_\mu - \frac{1}{8\pi G} * C_\mu \right)}$$

$$\boxed{f_{(1)\mu\nu} = \frac{16\pi G}{3K^2}\left(\tau_{\mu\nu} + \frac{1}{8\pi G K^2} * C_{\mu\nu} \right)}$$

C_μ and $C_{\mu\nu}$ are component of the

Cotton tensor:

$$C_{\mu\nu} = \eta_\mu{}^{\rho\sigma} \nabla_\rho \left(R_{\nu\sigma} - \frac{R}{4} g_{\nu\sigma} \right)$$

Longitudinal and transverse decomposition:

$$C_{\mu\nu} = \frac{3}{2} \frac{c u_\mu u_\nu}{k} + \frac{ck}{2} g_{\mu\nu} - \frac{c_{\mu\nu}}{k} + \frac{2u_{(\mu} c_{\nu)}}{k}$$

$\mathcal{E}_{uu} = 0$ and $\mathcal{E}_{ui} = 0$ ensures that

$$\nabla_\mu T^{\mu\nu} = 0$$

Flat limit

The advantage of the covariant NU gauge is that it allows for a vanishing cosmological contact limit in the same way as we did when taking the Carrollian limit of a relativistic theory. To do so it is convenient to work out the conformal boundary in the R-P gauge.

For the boundary

1) $g_{\mu\nu} dx^\mu dx^\nu = -K^2(\Omega du - b_i dx^i)^2 + a_{ij} dx^i dx^j$

2) $u_\mu = \frac{1}{\Omega}\partial u$ (Rest frame)

$\rightarrow \quad u_\mu u^\mu = -K^2$

and $u_\mu = -K^2(\Omega du - b_i dx^i)$

3) We have the following behaviour for the components of the $T^{\mu\nu}$:

$$\mathcal{E} = \sum_{m \geq 0} \mathcal{E}_{(m)} \kappa^{2m}$$

$$q^i = Q^i + \kappa^2 \Pi^i + \sum_{m \geq 2} \kappa^{2m} \Pi^i_{(m)}$$

$$\tau^{ij} = -\frac{\Sigma^{ij}}{\kappa^2} - \Xi^{ij} - \kappa^2 E^{ij} - \sum_{m \geq 2} \kappa^{2m} E^{ij}_{(m)}$$

$\kappa \to 0$ in the bulk

Order $\mathcal{O}(r^2)$:

$$\lim_{\kappa \to 0} r^2 g_{\mu\nu} dx^\mu dx^\nu = r^2 \left(0 \cdot du + a_{ij} dx^i dx^j \right)$$
$$= r^2 dl^2$$

* a flat limit makes the conformal boundary to become a conformal null geometry with degenerate metric

$$dl^2 = 0 \cdot du + d_{ij} dx^i dx^j$$

with Carroll vector

$$v = \frac{1}{\Omega} \partial_u$$

and clock form

$$\mu = -\Omega du - b_i dx^i \qquad \lim_{\kappa \to 0} \frac{u_\mu}{\kappa^2} = \mu$$

Order $\mathcal{O}(r)$:

$$\lim_{\kappa \to 0} \left(r A_\nu dx^\nu + r C_{\mu\nu} dx^\mu dx^\nu \right)$$

$$\boxed{= r\mu(2\varphi_i dx^i - \theta\mu) + r C_{ij} dx^i dx^j}$$

Here $C_{ij}(u, x)$ becomes an arbitrary function.

$$\sigma_{ij} = \xi_{ij} + \mathcal{O}(c^2)$$

$$\kappa^2 C_{ij} = -2\xi_{ij} \quad / \lim_{\kappa \to 0}$$

$$\Rightarrow \boxed{\xi_{ij} = 0} \quad \text{and } C_{ij} \text{ arbitrary}$$

$\xi_{ij} = 0$ implies that $a_{ij} = \lambda(\tau)\,\bar{a}_{ij}(x)$

Only in $d=2$!!

The new tensor is defined on a Carroll symmetric tensor on

$$\boxed{\hat{N}_{ij} = \frac{1}{\Omega}\hat{D}_u C_{ij}}$$

Order $\mathcal{O}(0)$:

$$\lim_{k\to 0}\left(\frac{2u_\mu dx^\mu}{k^2} + \frac{1}{k^4} S_{\mu\nu}\, dx^\mu dx^\nu\right)$$

$$g_{r\mu} \to g_{ri} = T$$
$$= -\Omega d\tau + b_i dx^i$$

from

$$\lim_{k\to 0}\frac{1}{k^4} S_{\mu\nu}\, dx^\mu dx^\nu \qquad a_{ij}\,dx^i dx^j$$

$$= \left(\frac{C_{kl} C^{kl}}{8} + {*\omega^2} \right) dl^2$$

$$- \hat{R}_2 T^2 - \hat{\varrho}_j C^j{}_i \, dx^i T - 2 * \hat{\varrho}_i * \omega \, dx^i T + * \omega * C_{ij} \, dy^i dx^j$$

Order $\mathcal{O}\left(\frac{1}{r}\right)$:

$$\lim_{K \to 0} \left(f_{(1)} \frac{u^2}{K^4} + \frac{2u}{K^2} f_{(1),i} \, dx^i + f_{(1),ij} \, dx^i dx^j \right)$$

$$= 8\pi G \, \mathcal{E}_{(0)} T^2 - \frac{4}{3} T \underbrace{\left(*\psi_i - 8\pi G \, \pi_i \right)}_{N_i} dx^i - \frac{16\pi G}{3} E_{ij} \, dx^i dx^j$$

The latter required the following condition to avoid divergences:

$$\lim_{K \to 0} \frac{u}{K^2} f_{(1),i} \, dx^i = \lim_{K \to 0} \frac{16\pi G}{3} \frac{T}{K^2} \left(q_i - \frac{1}{8\pi G} * C_i \right)$$

but

$C_i = \chi_i + k\,\Psi_i$

$S_{ij} = 0$!!

$\Rightarrow$

$$\frac{16\pi G}{3} M \lim_{k\to 0} \left(\frac{1}{k^2}\left(Q_i - \frac{1}{8\pi G}\chi_i\right) + \Pi_i - \frac{1}{8\pi G} {}^*\Psi_i \right)$$

$\Rightarrow$ $\boxed{Q_i = \frac{1}{8\pi G}\chi_i}$

Q_i is fixed through the geometry !!

From

$$\lim_{k\to 0} F(n)_{ij}\,dx^i dx^j = \frac{16\pi G}{3} \lim_{k\to 0} \left(\frac{1}{k^2} T_{ij} + \frac{1}{8\pi G k^4} {}^*C_{ij} \right)$$

but

$$C_{ij} = X^{ij} + k^2 \Psi^{ij}$$

$\Rightarrow$

$$\frac{16\pi G}{3}\lim_{k\to 0}\Big(\frac{1}{k^4}\Big(-\Sigma_{ij}+\frac{1}{8\pi G}{*X_{ij}}\Big)$$

$$+\frac{1}{k^2}\Big(-\mho_{ij}+\frac{1}{8\pi G}\Psi^{ij}\Big)-E_{ij}\Big)$$

$$\Rightarrow \boxed{\Sigma_{ij}=\frac{1}{8\pi G}*X_{ij}} \qquad \boxed{\mho_{ij}=\frac{1}{8\pi G}*\Psi_{ij}}$$

$$ds^2 = T\Big[2dr+(2r\varphi_i-2*\hat{\eta}_i*\omega-\hat{\eta}_j C^j{}_i)dx^i$$

$$-(r\theta+\hat{h})T\Big]+\Big(r^2+*\omega^2+\frac{C_{kl}C^{kl}}{8}\Big)d\ell^2$$

$$+(rC_{ij}+*\omega*C_{ij})dx^i dx^j$$

$$+\frac{1}{r}\Big(8\pi G\,\mathcal{E}_{(0)}T^2-\frac{4}{3}T N_i dx^i-\frac{16\pi G}{3}E_{ij}dx^i dx^j\Big)$$

$$+O\Big(\frac{1}{r^2}\Big)$$

Einstein equations requires the following:

$$\boxed{\begin{gathered}\frac{1}{\Omega}\hat{D}_T M - \frac{1}{2}\hat{D}_i \chi^i = f(N_{ij}, C_{ij}) \\[2em] \frac{1}{\Omega}\hat{D}_T N^i - \hat{D}^i M + *\hat{D}^i N = f^i(N_{ij}, C_{ij})\end{gathered}}$$

with

$$M = 4\pi G \mathcal{E}_{(0)} - \frac{1}{8} C^{ij} \hat{N}_{ij} \Big\} \text{ covariant Bondi mass}$$

$$N^i = *\varphi^i - N^i \Big\} \text{ angular momentum aspect}$$

$$N = \frac{1}{2} C_{(0)} - \frac{1}{4} \hat{D}_i \hat{D}_j * C^{ij} \Big\} \text{ magnetic mass}$$

$$C = C_{(0)} + k^2 C_{(\ldots)}$$

A Ricci flat spacetime is reconstructed in terms of Carrollian boundary data that correspond to:

1) Carroll geometry at null infinity

2) Carroll momenta $\mathcal{E}_{(0)}$, Π^i and E_{ij}

3) Dynamic Shear C_{ij}

An infinite set of Carroll data appear if we go deeper in the asymptotic expansion which corresponds to the subleading term of $T^{\mu\nu}$: $\mathcal{E}_{(n)}$, $\Pi^i_{(n)}$, $E_{(n)ij}$

These function are completely arbitrary and are restric to satisfy some balance equation (Carrollian)