Lectures on Duality in Condensed Matter Physics

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Outline

* Electromagnetic Duality and the duality of forces
* Ising Models: Kramers - Wannier dualities
* Gauge theory and duality
* Vortices and Monopoles
* Particle - Vortex duality
* Boson - Fermion mappings as duality
* Duality and the Fractional Quantum Hall Effect

Electromagnetic Duality (Dirac, 1931)

$$\vec{\nabla}\cdot\vec{E} = \rho, \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \vec{J}, \qquad \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$$

* EM Duality: $\vec{E} \longleftrightarrow \vec{B}$ (if $\rho = 0$ and $\vec{J} = 0$)

* electric charge e $\longleftrightarrow$ magnetic monopole m

* Dirac quantization: $em = 2\pi$

* $\partial_\mu F^{\mu\nu} = j^\nu$; $\partial^\mu F^*_{\mu\nu} = 0$; $F^*_{\mu\nu} \equiv \frac{1}{2}\varepsilon_{\mu\nu\lambda\rho}F^{\lambda\rho}$

  field tensor, charge current, Bianchi Identity, dual field

* Bianchi Identity $\Longleftrightarrow$ absence of magnetic monopoles

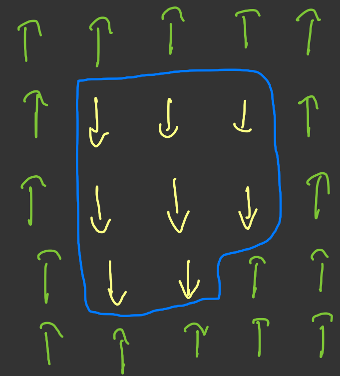
Duality of Forms

* Electromagnetic duality is geometric
* In Differential Geometry a vector field A_μ is a 1-form
* The field tensor $F_{\mu\nu}$ is a 2-form
* The dual field $F^*_{\mu\nu}$ is also a 2-form
* In d-dimensions a p-form (antisymmetric tensor of rank p) is dual to a $d-p$ form
* In $d=4$ dimensions a 2-form is dual to a 2-form
* In $d=2$ dimensions a 1-form is dual to a 1-form

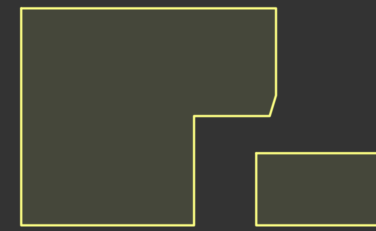
$$\partial_\mu \phi = \varepsilon_{\mu\nu} \partial^\nu \psi$$ Cauchy-Riemann!

Duality in Ising Models

* 2D: the Ising model is (1941) self-dual (Kramers - Wannier)
* Low T: Z is an expansion in closed domain wall loops; weight $\sim e^{-2/T} \times$ length
* High T: Z is an expansion in closed loops weight $\sim \tanh(1/T) \times$ length



domain walls (low T)



high T expansion diagrams

* maps high T $\Longleftrightarrow$ low T
  disorder $\Longleftrightarrow$ order

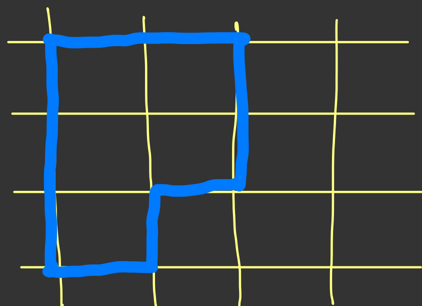
$$Z = \sum_{[\sigma]} e^{\frac{1}{T} \sum_{\langle x, x' \rangle} \sigma(x)\sigma(x')}$$

* self-dual: $e^{-2/T_c} = \tanh(1/T_c)$

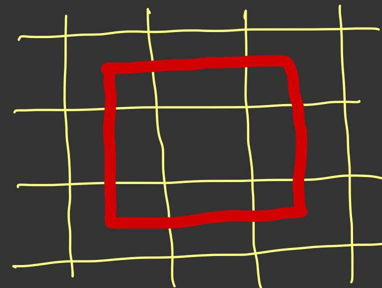
$\Longrightarrow \quad \frac{1}{T_c} = \frac{1}{2} \ln(\sqrt{2}+1)$ (Onsager, 1944)

$$Z_{DW}\left[e^{-\frac{2}{T}}\right] = Z_{loops}\left[\tanh \frac{1}{T}\right]$$

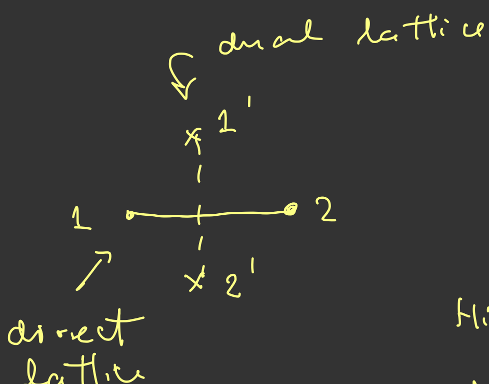
* The high-T_c loops live on the direct lattice
* The low-T_c loops (domain walls) live on the <u>dual lattice</u>



direct loops



dual loops (domain walls)



In 2d $\left\{ \begin{array}{l} \text{links are dual to links} \\ \text{sites are dual to plaquettes} \end{array} \right.$

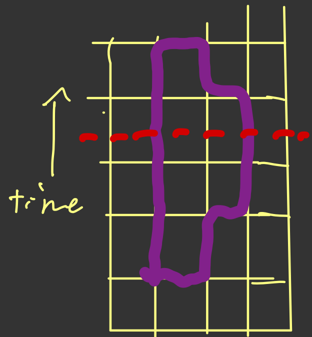
This is the same as the duality of forms

High T: $$Z = \sum_{\{loops\}} \left(\tanh \frac{1}{T}\right)^{L(loops)}$$

Low T: $$Z = \sum_{\substack{\text{domain}\\ \text{walls}}} \left(e^{-\frac{2}{T}}\right)^{L(DW)}$$

Field Theory Interpretation

* We can regard the P.F. Z as a sum over histories of spin configurations from one row to the next row
* Path integral in a discretized imaginary time
* High T expansion loops $\iff$ processes in which pairs of particles are created and destroyed



* Low T expansion loops $\iff$ processes in which pairs of domain walls are created and destroyed
* Analog of the (imaginary) time evolution operator is the transfer matrix
* The classical d-dimensional Ising Model is equivalent to a quantum Ising Model in d-1 dimensions

The Quantum Ising Model (EF & L. Susskind)

$d = 1$

$$H = -\sum_j \sigma_1(j) - \lambda \sum_j \sigma_3(j)\,\sigma_3(j+1)$$

PBC's

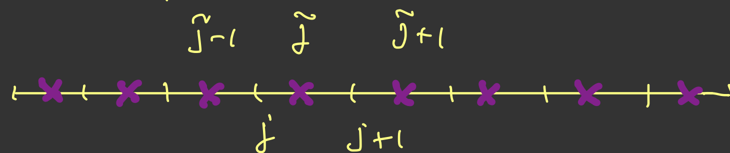
(a) λ small $\iff$ T high $\iff$ disorder

(b) λ large $\iff$ T low $\iff$ order

Global $\mathbb{Z}_2$ symmetry: $Q = \prod_j \sigma_1(j)$; $[Q, H] = 0$

flipping all spins

Duality:



dual lattice: midpoints

$$\tau_1(\tilde{j}) = \sigma_3(j)\,\sigma_3(j+1) \; ; \quad \tau_3(\tilde{j}) = \prod_{n \le j} \sigma_1(n)$$

creates domain walls

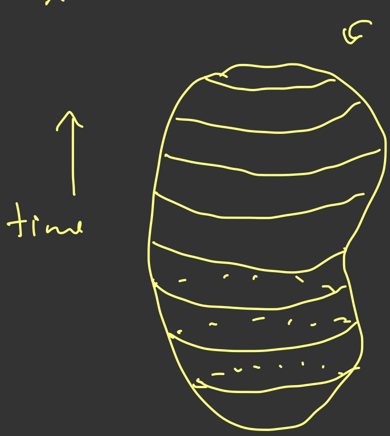
$$\tau_3(\tilde{j}-1)\,\tau_3(\tilde{j}) = \sigma_1(j) \; ; \quad \tau_1^2 = \tau_3^2 = 1, \quad \{\tau_1, \tau_3\} = 0$$

$$H = -\sum_{\tilde{j}} \tau_3(\tilde{j})\,\tau_3(\tilde{j}+1) - \lambda \sum_{\tilde{j}} \tau_1(\tilde{j})$$

duality $\lambda \longleftrightarrow \frac{1}{\lambda}$ $\Longrightarrow$ self duality $\lambda = 1$ (Onsager)

Dual of the D = 3 Classical Ising Model

* The high T expansion is a sum over loop configurations
* The low T expansion is a sum over domain wall configs.
* In D = 3 the domain walls are closed surfaces



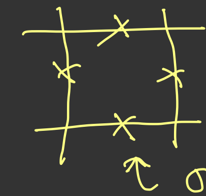
A domain wall configs. can be pictured as the time evolution of a closed string on the dual lattice

$\Rightarrow$ the dual model at low $\tilde{T}$ is a sum over loops and at high $\tilde{T}$ is a sum over closed surfaces

The D=3 Ising Gauge Theory (Wegner 1971)

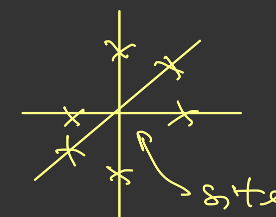
$\begin{cases} Z_2 \text{ gauge fields on the links } \{\sigma_\mu\} \\ \text{Interactions on plaquettes} \end{cases}$ $\qquad \sigma_\mu = \pm 1$

$$Z = \sum_{\{\sigma_\mu\}} \exp\left(\frac{1}{T}\sum_{\text{plaquettes}} \sigma\sigma\sigma\sigma\right)$$



plaquette of a cube

Gauge invariance : flip all Z_2 gauge fields that share a site

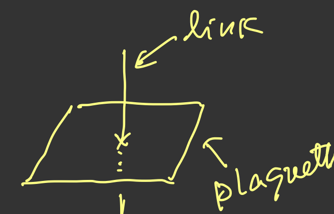


High T $\Rightarrow$ sum over closed surfaces with a weight $\left(\tanh\frac{1}{T}\right)^{\text{surface}}$

Low T $\Rightarrow$ sum over closed loops of the dual lattice

This is the dual of the D=3 Ising Model

In D=3 links are dual to plaquettes (1 form $\Leftrightarrow$ 2 form)



Observables

3D Ising Model

Correlator $\langle \sigma(x)\, \sigma(x') \rangle$

* high T $\sim e^{-|x-x'|/\xi}$

disorder

* low T $\sim |\langle \sigma \rangle|^2 + e^{-|x-x'|/\xi}$

Long range order

Wilson Loop: creates an open domain wall (defect)

Area law in the ordered phase

Perimeter law in the disordered phase

3D $\mathbb{Z}_2$ gauge theory

Wilson loop $\langle \prod_\gamma \sigma \rangle$; $\partial\Sigma = \gamma$

γ: closed loop

* high T^* $\sim \exp - \mathrm{Area}(\Sigma)$

confinement

* low T^* $\sim \exp(-\mathrm{length}(\gamma))$

deconfinement

Correlator: creates an open $\mathbb{Z}_2$ flux tube ending at two "monopoles"

Confinement $\Leftrightarrow$ monopole condensation

(EF & L. Susskind, '78) Quantum Version (2+1 dimensions)

Ising Model

$$H = -\sum_{\vec{r}} \sigma_1(\vec{r}) - \lambda \sum_{\langle \vec{r}, \vec{r}\,' \rangle} \sigma_3(\vec{r})\,\sigma_3(\vec{r}\,')$$

Global $\mathbb{Z}_2$ symmetry

$$Q = \prod_{\text{sites}} \sigma_1(\text{sites})$$

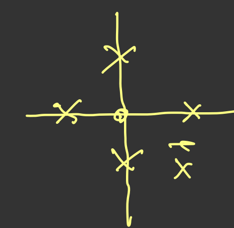
$\mathbb{Z}_2$ Gauge Theory

$$H = -\sum_{\text{links}} \sigma_1(\text{link}) - g \sum_{\text{plaquettes}} \sigma_3\sigma_3\sigma_3\sigma_2$$

Local (gauge) $\mathbb{Z}_2$ symmetry

$$Q(\vec{x}) = \prod_{\substack{\text{links} \\ \text{that} \\ \text{share } \vec{x}}} \sigma_1(\text{links})$$

("star")



$$[Q(\vec{x}), Q(\vec{x}\,')] = 0$$

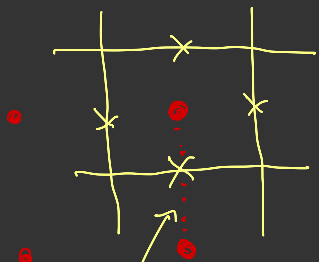
$$[Q(\vec{x}), H] = 0$$

Gauge Invariant states

$$Q(\vec{x})\,|\text{Phys}\rangle = |\text{Phys}\rangle$$

Gauss Law

Quantum version of Duality



dual link

$$\prod_{\text{plaquette}} \sigma_3 = \tau_1 \text{ (dual site)}$$

$$\sigma_1 (\text{link}) = \tau_3 \tau_3 \quad \text{dual link}$$

$$\prod_{\text{star}} \sigma_1 = 1$$

$\Rightarrow$ The gauge invariant sector ($Q(\vec{x}) = 1$) of the $\mathbb{Z}$ gauge theory with coupling g maps onto the 2+1 dim. Ising Model with coupling $\lambda = \frac{1}{g}$

$$H = -\sum_{\text{links}} \sigma_1 - g \sum_{\text{plaquettes}} \sigma_3\sigma_3\sigma_3\sigma_3 \xleftrightarrow[\text{duality}]{} H = -\sum \tau_3\tau_3 - \lambda \sum \tau_1$$

sites are dual to plaquettes, links are dual to links

Physical Picture

Confined phase $(g < g_c)$ (use σ_1 eigenstates)

* $|Gnd\rangle = \sum_{\text{loops}} \# \; |$  $\rangle$

  loops created by the plaquette operator

  $\mathbb{Z}_2$ electric loops are small for g small

* At g_c the loops proliferate

* For $g \gg g_c$ we approximate $H = -\sum_{\text{plaquettes}} \sigma_3 \sigma_3 \sigma_3 \sigma_3$

  $Q(x) = 1$ ("Toric Code")

* On a torus it has a 4-fold degeneracy (Topological Phase)

* The dual is $H_{\text{Ising}} = -\sum_{\text{sites}} \tau_1 \Rightarrow$ disordered phase (unique state) (Kitaev, 1997)

Vortices and Monopoles

We will discuss models with a (compact) $U(1)$ symmetry

Complex scalar $\phi(x) = |\phi(x)|\, e^{i\theta(x)}$ (θ defined mod 2π)

Order parameter of an XY classical spin, superfluid or an incommensurate CDW

Global symmetry $\phi(x) \to \phi(x)\, e^{i\alpha} \iff \theta(x) \to \theta(x) + \alpha$

Ordered phase (T low) $|\phi(x)| \approx \phi_0$

$$Z \approx \int \mathcal{D}\theta(x)\, \exp\left(-\int d^2x\, \frac{1}{2g} (\vec{\nabla}\theta)^2\right) \qquad \theta \cong \theta + 2\pi$$

$$g = T / J|\phi_0|^2 \quad , \quad K = J|\phi_0|^2 = \text{stiffness}$$

Vortices

C: closed oriented path

Total change of phase : $\frac{(\Delta\Theta)_C}{2\pi}$
on the closed path C

$$\frac{(\Delta\Theta)_C}{2\pi} = \frac{1}{2\pi}\oint_C d\vec{x}\cdot\vec{\nabla}\Theta(x) = i\int_0^{2\pi}\frac{d\varphi}{2\pi}\, e^{i\Theta(\varphi)}\,\partial_\varphi e^{-i\Theta(\varphi)} \equiv m$$

vorticity

m: topological invariant under smooth deformations of C

$\Theta(x)$ is a map of $C \to$ phase field $e^{i\Theta}$

$$\Rightarrow \quad \underset{\substack{\uparrow\\ \text{base}}}{\Theta(x) : S_1} \longrightarrow \underset{\substack{\uparrow\\ \text{target}}}{S_1} \qquad \pi_{\underset{\text{base}}{1}}(\underset{\text{target}}{S_1}) \cong \mathbb{Z}$$

homotopy class

m : winding number

<u>Superfluid current</u> $j_\mu = \partial_\mu \theta$

<u>Vorticity</u> $\omega(x) = \varepsilon_{\mu\nu} \partial_\mu j_\nu = \varepsilon_{\mu\nu} \partial_\mu \partial_\nu \theta(x)$

$\uparrow$ Levi-Civita

$\Rightarrow$ $\theta(x)$ has a branch cut singularity accross which it jumps by $2\pi m$

* set of vortices at locations $\{\vec{x}_j\}$ with topological charges $\{m_j\}$

$$\Rightarrow \quad \omega(\vec{x}) = 2\pi \sum_j m_j \, \delta^2(\vec{x} - \vec{x}_j)$$

$z = x_1 + i x_2$

$$\equiv 2\pi \sum_j m_j \, \mathrm{Im} \ln (z - z_j)$$

Define ϑ, the Cauchy-Riemann dual $\partial_\mu \vartheta = \varepsilon_{\mu\nu} \partial_\nu \theta$

$$\Rightarrow -\nabla^2 \vartheta = \omega(x)$$

$$\Rightarrow \quad \vartheta(\vec{x}) = \int d^2y \; G(|\vec{x}-\vec{y}|)\, \omega(\vec{y})$$

$$-\nabla^2 G(\vec{x}-\vec{y}) = \delta^2(\vec{x}-\vec{y}) \qquad \text{Green Function}$$

$$G(|\vec{x}-\vec{y}|) = \frac{1}{2\pi} \ln\left(\frac{a}{|\vec{x}-\vec{y}|}\right)$$

UV cutoff s.t. $G(|x-y|) = 0$ if $|\vec{x}-\vec{y}| < a$

<u>Energy of the Config:</u>

$$E[\theta] = \frac{J\Phi_0^2}{2}\int d^2x \,(\vec{\nabla}\theta)^2 = \frac{J\Phi_0^2}{2}\int d^2x \int d^2y \; \omega(x)\, G(x-y)\, \omega(y)$$

$$= 2\pi J \Phi_0^2 \sum_{j<k} m_j m_k \ln\left(\frac{a}{|x_j - x_k|}\right)$$

which is IR divergent <u>unless</u> $\sum_j m_j = 0$ (zero total vorticity)

$$Z_{XY} \approx Z_{\substack{\text{Coulomb}\\ \text{Gas}}} = \sum_{\{m_j\}}{}' \exp\left(-\frac{2\pi |\Phi_0|^2 J}{T}\sum_{j<k} m_j m_k \ln\left(\frac{a}{|x_j - x_k|}\right)\right)$$

Kosterlitz - Thouless Transition

At low T the vortices are bound in neutral pairs

The free energy of a vortex is

$$F_{vortex} = E_{vortex} - T\, S_{vortex}$$

$$E_{vortex} = \pi J |\phi_0|^2 \ln\left(\frac{L}{a}\right) \leftarrow \text{Energy} \quad (L: \text{linear size of the system})$$

$$S_{vortex} = \ln\left(\frac{L}{a}\right)^2 \leftarrow \text{Entropy}$$

$$F_{vortex}(T_{KT}) = 0 \iff T_{KT} = \frac{\pi}{2} J \phi_0^2$$

$T < T_{KT}$ vortices are suppressed

$T > T_{KT}$ vortices proliferate

Alternative Picture

Let A_μ be a background $U(1)$ gauge field

$$Z[A] = \int \mathcal{D}\theta \; e^{-\frac{1}{2g}\int d^2x \, (\partial_\mu \theta - A_\mu)^2}$$

Let A_μ represent a vortex field $\varepsilon_{\mu\nu}\partial_\mu A_\nu = \omega(x)$

Hubbard-Stratonovich:

$$Z[A] = \int \mathcal{D}\theta \, \mathcal{D}a_\mu \; e^{-\frac{g}{2}\int d^2x \, a_\mu^2 + i\int d^2x \, a_\mu(\partial_\mu\theta - A_\mu)} \Rightarrow a_\mu = \varepsilon_{\mu\nu}\partial_\nu\vartheta$$

$$Z[A] = \# \int \mathcal{D}\vartheta \; e^{-\frac{g}{2}\int d^2x \, (\partial_\mu\vartheta)^2 + i\int d^2x \, \vartheta\,\omega}$$

with $\int d^2x \, \omega = 0$

Duality: $\theta \longleftrightarrow \vartheta$ and $g \longleftrightarrow \frac{1}{g}$

Summing over vortices with core energy $u m^2$

$$Z = \sum_{\{m_j\}}{}' \int \mathcal{D}\vartheta \, \exp\left[-\frac{g}{2}\int d^2x \, (\partial_\mu \vartheta)^2 + i \sum_j 2\pi m_j \vartheta(x_j) - \frac{u}{T}\sum_j m_j^2\right]$$

$z = e^{-u/T} \ll 1 \Rightarrow$ only $m_j = 0, \pm 1$ contribute

$$\Rightarrow Z = \int \mathcal{D}\vartheta \, \exp\left[-\int d^2x \left[\frac{g}{2}(\partial_\mu \vartheta)^2 - v \cos(2\pi\vartheta)\right]\right]$$ Sine-Gordon

$v = 2z/a^2$

vortex correlator: $\langle e^{2\pi i \vartheta(x)} e^{-2\pi i \vartheta(y)} \rangle = \dfrac{\text{const.}}{|x-y|^{2\pi/g}}$

$\Rightarrow$ scaling dimension $\Delta_{\text{vortex}} = \pi/g$

$\Rightarrow$ vortices are <u>relevant</u> if $\Delta_{\text{vortex}} < d = 2 \Rightarrow g_c = \dfrac{\pi}{2}$

$\Rightarrow$ KT transition

* $\langle e^{i\theta(x)} e^{-i\theta(y)} \rangle = \dfrac{\text{const}}{|x-y|^{g/2\pi}}$ $\Rightarrow$ $\dfrac{g}{2\pi} = \dfrac{T}{2\pi J \phi_0^2} \leq \dfrac{1}{4}$ for all $T < T_{KT}$

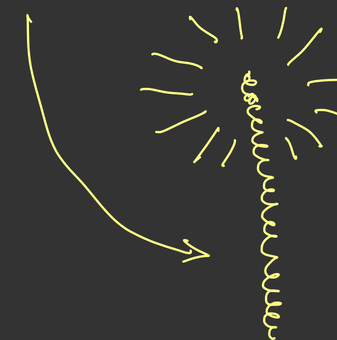
Power law decay

Magnetic monopoles in Compact QED $d=3$

(Euclidian spacetime)

Dirac monopole $\quad B_i(x) = \frac{q}{2}\frac{x_i}{2} - 2\pi q\, \delta_{i,3}\, \delta(x_1)\, \delta(x_2)\, \Theta(-x_3)$

Dirac string



Lattice model (Polyakov, 1977)

$$Z = \prod_{\text{links}} \int_0^{2\pi} \frac{dA_\mu}{2\pi} \exp\left(\frac{1}{4e^2} \sum_{\text{plaquettes}} \cos F_{\mu\nu}\right)$$

$F_{\mu\nu}$: flux through a plaquette

$F_{\mu\nu} = \Delta_\mu A_\nu - \Delta_\nu A_\mu$

Gauge invariance: $\quad A_\mu \to A_\mu + \Delta_\mu \Phi(x)$

Periodicity: $\quad A_\mu \longrightarrow A_\mu + 2\pi l_\mu \;, \; l_\mu \in \mathbb{Z}$

$$\prod_{\substack{\text{cube}\\ \text{faces}}} e^{iF_{\mu\nu}} = 1 \qquad (\text{Bianchi Identity})$$

$\Rightarrow$ allows for monopoles with $q \in \mathbb{Z}$

We will follow the same approach we used with vortices

$\Rightarrow B_{\mu\nu}$ is a background 2-form field $(B_{\mu\nu} = -B_{\nu\mu})$
(Kalb-Ramond)

$$Z[B_\mu] = \int \mathcal{D}A_\mu \, \exp\left(-\frac{1}{4e^2}\int d^2x \left(F_{\mu\nu} - B_{\mu\nu}\right)^2\right)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \, ; \qquad A_\mu \to A_\mu + \partial_\mu \Phi \, ; \; B_{\mu\nu} \to B_{\mu\nu}$$

we also have $\quad A_\mu \to A_\mu + a_\mu \, , \quad B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu a_\nu - \partial_\nu a_\mu$
(2-form gauge transf.)

Monopole density: $\quad \mathcal{M}(x) = 2\pi \sum_j m_j \, \delta^3(x - x_j)$

$$\mathcal{M}(x) = \frac{1}{2} \varepsilon_{\mu\nu\lambda} \partial_\mu B_{\nu\lambda}$$

$$Z[B] = \int \mathcal{D}A_\mu \mathcal{D}b_{\mu\nu} \, \exp\left(-\frac{e^2}{2}\int d^3x \, b_{\mu\nu}^2 + i\int d^3x \, \frac{1}{2} b_{\mu\nu}\left(F_{\mu\nu} - B_{\mu\nu}\right)\right)$$

$\Rightarrow \partial_\nu b_{\mu\nu} = 0 \Rightarrow b_{\mu\nu} = \varepsilon_{\mu\nu\lambda}\partial_\lambda \vartheta \qquad$ (compact scalar)

invariant under $\vartheta \to \vartheta + \alpha \quad$ (α: const mod 2π)

$$\Rightarrow Z[B] = \int \mathcal{D}\vartheta \, \exp\left(-\frac{e^2}{2}\int d^3x \, (\partial_\mu \vartheta)^2 + 2\pi i \sum_j m_j \vartheta(j)\right)$$

$$\Rightarrow Z = \sum_{\{m_j\}}{}' Z[m_j] \equiv \int \mathcal{D}\vartheta \, \exp\left(-\int d^3x \left(\frac{e^2}{2}(\partial_\mu \vartheta)^2 - v \cos 2\pi\vartheta\right)\right)$$

sine-Gordon! but $\underline{d=3}$

$v = 2\exp(-u)/a^3$ (u: core energy)

Monopole correlator

$$\langle e^{i 2\pi \vartheta(x)} e^{-i 2\pi \vartheta(y)} \rangle = \exp\left(\frac{\pi}{2e^2}\left[\frac{1}{R} - \frac{1}{a}\right]\right) \quad R = |x-y|$$

$$\xrightarrow[R \to \infty]{} \text{const}$$

$\Rightarrow$ Monopoles proliferate for $\underline{\text{all}}$ $e^2 \neq 0$

$\Rightarrow$ In $d=3$ the energy $< \infty$ but the entropy $\sim \ln\left(\frac{L}{a}\right)^3 \to \infty$

$\Rightarrow$ confinement by monopole condensation

Wilson loop: $W_\gamma = \langle e^{i \oint_\gamma dx_\mu A_\mu} \rangle$ has an area law $\Rightarrow$ confinement

(Polyakov, 1977)

Higgs, Confinement and Topology $(d=3)$

Consider a theory of complex order parameter of charge $n \in \mathbb{Z}$ coupled to a dynamical $U(1)$ (compact) gauge field

Order Parameter field $e^{i\Theta(x)}$, A_μ gauge field

$n=2$ is (with some caveats) the case of a superconductor

Lattice model: $\Theta(x)$ on sites and A_μ on links

$$Z = \prod_{\text{sites}} \int_0^{2\pi} \frac{d\Theta}{2\pi} \prod_{\text{links}} \int_0^{2\pi} \frac{dA_\mu}{2\pi} \exp\left(S(\Theta, A_\mu)\right)$$

$$S = \beta \sum_{\text{links}} \cos\left(\Delta_\mu \Theta - n A_\mu\right) + \frac{1}{g^2} \sum_{\text{plaquettes}} \cos\left(F_{\mu\nu}\right)$$

$g \to 0 \Rightarrow F_{\mu\nu} \to 0 \pmod{2\pi} \Rightarrow$ 3D XY model $\Rightarrow$ "Higgs" for β large

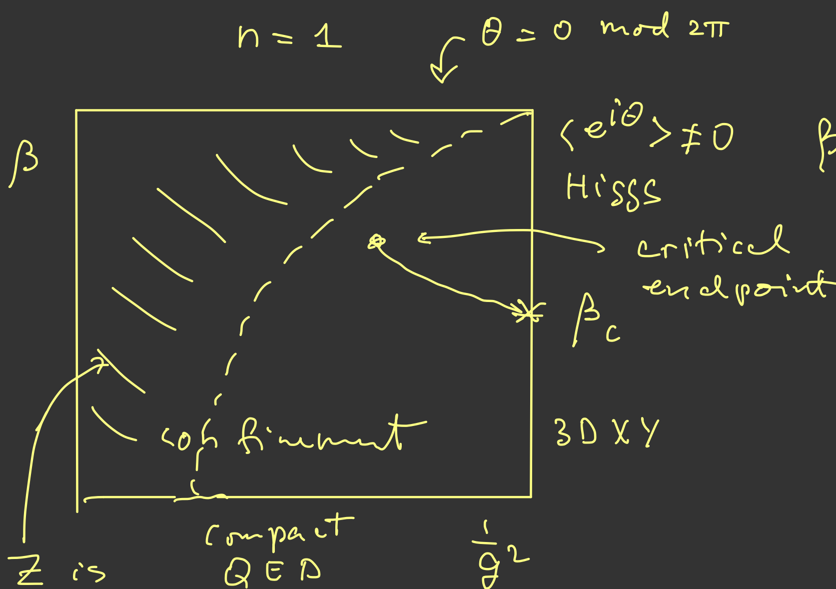
$\beta \to 0 \Rightarrow$ Polyakov's QED $\Rightarrow$ confinement

Q1: How are the Higgs and confinement limits related?

Q2: Are they different phases

Answer: it depends on n (EF & S. Shenker 1979)

$n = 1$

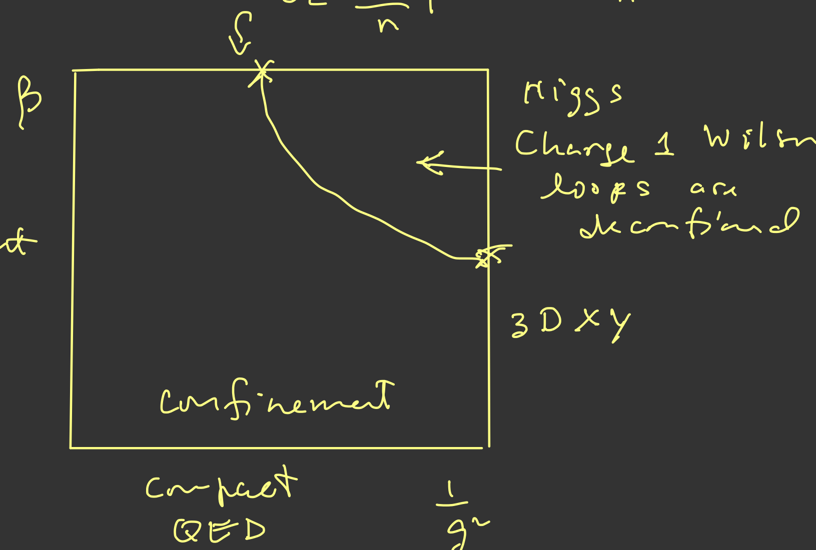


Z is analytic

No phase transition

Higgs and Confinement are the same phase

$\theta = \frac{2\pi}{n} p \iff \mathbb{Z}_n$ spin model



Consider the deconfined phase $n > 1$

$$Z = \int \mathcal{D}\theta\, \mathcal{D}A_\mu \exp\left(-\int d^3x \left[\frac{\beta}{2}(\partial_\mu \theta - n A_\mu)^2 - \frac{1}{4g^2} F_{\mu\nu}^2\right]\right)$$

for $\beta \gg 1$

Hubbard - Stratonovich

$$Z = \int \mathcal{D}\theta\, \mathcal{D}A_\mu\, \mathcal{D}a_\mu\, e^{-\int d^3x \frac{1}{2\beta} a_\mu^2 + i\int a_\mu(\partial_\mu \theta - n A_\mu) - \int \frac{1}{4g^2} F_{\mu\nu}^2}$$

Integrate θ out $\Rightarrow$ $\partial_\mu a_\mu = 0 \Rightarrow a_\mu = \varepsilon_{\mu\nu\lambda}\partial_\nu b_\lambda$

$$Z = \int \mathcal{D}b_\mu\, \mathcal{D}A_\mu \exp\left(i n \int d^3x\, A_\mu \varepsilon_{\mu\nu\lambda}\partial_\nu b_\lambda - \int d^3x \frac{1}{4\beta} f_{\mu\nu}^2 - \int d^3x \frac{1}{4g^2} F_{\mu\nu}^2\right)$$

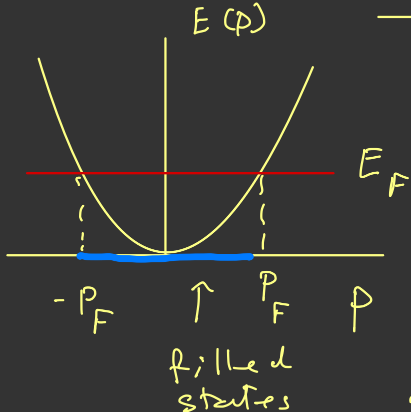
$\beta \to \infty$ and $g \to 0$ only the "BF" term survives

This is a topological term $\Rightarrow$ The deconfined phase is topological $\neq$ Higgs

For $n = 2 \Rightarrow$ Toric Code

There is never a Higgs phase

Fermions in one-dimension



$$E(p) \simeq v_F(p-p_F) - v_F(p+p_F) + \cdots$$

$$\Psi(x) \simeq \psi_R(x)\, e^{ip_F x} + \psi_L(x)\, e^{-ip_F x}$$

density $\rho(x) = \Psi^\dagger(x)\Psi(x)$

$$\simeq \bar{\rho} + \psi_R^\dagger \psi_R + \psi_L^\dagger \psi_L + \psi_R^\dagger \psi_L\, e^{i2p_F x} + \psi_L^\dagger \psi_R\, e^{-i2p_F x}$$

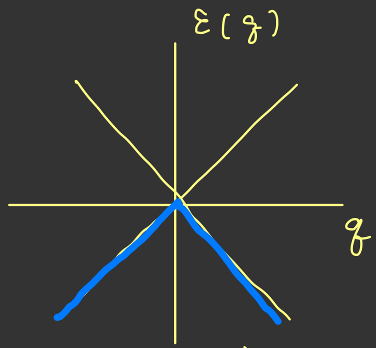
slowly varying density: $j_0 \equiv \psi_R^\dagger \psi_R + \psi_L^\dagger \psi_L \equiv \rho_R + \rho_L$

slowly varying current: $j_1 \equiv \psi_R^\dagger \psi_R - \psi_L^\dagger \psi_L \equiv \rho_R - \rho_L$

$\psi_R^\dagger(x)\psi_L(x) \equiv$ Order parameter of a CDW with $Q = 2p_F$; complex field

If $\langle \psi_R^\dagger \psi_L + \psi_L^\dagger \psi_R \rangle \neq 0 \implies$ energy gap at $p = \pm p_F$

The density, current and the CDW O.P. are invariant under the global U(1) gauge transformation $\Psi(x) \to \Psi'(x) = e^{i\theta}\Psi(x)$



filled Dirac sea

$q = p - p_F$ etc. $\quad \psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}$ Dirac spinor

$$\mathcal{H} = -v_F \left(\psi_R^+ i\partial_x \psi_R - v_F \psi_L^+ i\partial_x \psi_L \right)$$

2×2 Dirac γ-matrices

$$\gamma_0 = \sigma_1, \quad \gamma_1 = -i\sigma_2, \quad \gamma_5 = \sigma_3, \quad \{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}; \; g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Dirac Lagrangian density $\mathcal{L} = \bar{\Psi}\, i\gamma_\mu \partial^\mu \psi \;;\; \bar{\Psi} = \psi^+\gamma_0$

It has two continuous symmetries

* $\psi(x) \longrightarrow \psi'(x) = e^{i\theta}\psi(x) \Rightarrow \psi_R' = e^{i\theta}\psi_R, \; \psi_L' = e^{i\theta}\psi_L$ (gauge)
* $\psi(x) \rightarrow \psi'(x) = e^{i\gamma_5\theta}\psi(x) \Rightarrow \psi_R' = e^{+i\theta}\psi_R, \; \psi_L' = e^{-i\theta}\psi_L$ (chiral)

Q: How come we have <u>two</u> symmetries?

Chiral Anomaly

* The massless Dirac theory has two global symmetries whereas the microscopic model has only one : gauge invariance.
* In the massless Dirac theory ψ_R and ψ_L are separately conserved
* Two currents : the gauge current $j_\mu = \bar{\psi}\gamma_\mu \psi$, $\partial_\mu j^\mu = 0$ and the chiral current $j^5_\mu = \bar{\psi}\gamma_\mu \gamma_5 \psi$; $\partial^\mu j^5_\mu = 0$

$$j_\mu = (\rho_R + \rho_L , \rho_R - \rho_L) \quad , \quad j^5_\mu = (\rho_L - \rho_R , \rho_R + \rho_L)$$

* If we couple the theory to an uniform electric field E

$\Rightarrow \quad \dfrac{dN_R}{dt} = \dfrac{e}{2\pi} E$ and $\dfrac{dN_L}{dt} = -\dfrac{e}{2\pi} E \Rightarrow \dfrac{dQ}{dt} = 0 \Rightarrow$ electric charge is conserved

But $\dfrac{dQ_5}{dt} = \dfrac{e}{\pi} E \Rightarrow$ chiral charge is not conserved

(following H. Nielsen and Y. Ninomiya 1982)

This is the <u>chiral anomaly</u>

Chiral Anomaly and Bosonization

The normal-ordered gauge charge density j_0 and current j_1 satisfy the equal-time commutator

Mattis & Lieb, 1965
Luther & Emery, 1974
S. Coleman, 1975
E. Witten, 1978

$$[j_0(x_1), j_1(x_1')] = -\frac{i}{\pi}\partial_1 \delta(x_1 - x_1')$$

$$[j_0(x_1), j_0(x_1')] = [j_1(x_1), j_1(x_1')] = 0$$

Let ϕ be a real scalar field and Π the conjugate momentum

$$\Rightarrow [\phi(x), \Pi(x')] = i\delta(x - x')$$

$\Rightarrow$ we identify $\quad j_0(x) \equiv \frac{1}{\sqrt{\pi}}\partial_1\phi \quad , \quad j_1(x) \equiv -\frac{1}{\sqrt{\pi}}\Pi(x) = -\frac{1}{\sqrt{\pi}}\partial_0\phi$

$$\Rightarrow j_\mu(x) = \frac{1}{\sqrt{\pi}}\varepsilon_{\mu\nu}\partial^\nu\phi \quad (\text{duality!}) \Longleftrightarrow \partial^\mu j_\mu = 0 \quad \checkmark$$

But $j^5_\mu = \varepsilon_{\mu\nu} j^\nu \Rightarrow \partial^\mu j^5_\mu = -\frac{1}{\sqrt{\pi}}\partial^2\phi \Rightarrow \partial^\mu j^5_\mu = 0 \Longleftrightarrow \phi$ is a free field!

$$\mathcal{L} = \bar{\Psi} i\gamma^\mu\partial_\mu\Psi \longleftrightarrow \mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2$$

Coupling to a gauge field

$$\mathcal{L} = \bar{\Psi} i \gamma^\mu \partial_\mu \Psi - e A^\mu \bar{\Psi} \gamma_\mu \Psi$$

In the bosonic theory

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{e}{\sqrt{\pi}} A^\mu \varepsilon_{\mu\nu} \partial^\nu \phi$$

$\Rightarrow$ Eq. of motion $\quad -\partial^2 \phi = \frac{e}{\sqrt{\pi}} \varepsilon_{\mu\nu} \partial^\nu A^\mu \equiv \frac{e}{\sqrt{\pi}} F^*$ (dual "tensor")

$\Rightarrow \quad \partial^\mu j^5_\mu = -\frac{1}{\sqrt{\pi}} \partial^2 \phi = \frac{e}{2\pi} F^*$ chiral anomaly!

Total fermion # $N_F \in \mathbb{Z}$

$$N_F = \int_0^L dx_1 \, j_0(x_0, x_1) = \frac{1}{\sqrt{\pi}} \int_0^L dx_1 \, \partial_1 \phi(x_0, x_1) = \frac{1}{\sqrt{\pi}} \left(\phi(x_0, L) - \phi(x_0, 0) \right)$$

$\Rightarrow \phi(x_1 + L) = \phi(x_1) + \sqrt{\pi} N_F \quad \Rightarrow \quad \phi$ is a compactified scalar

$\phi(x+L) = \phi(x) + 2\pi R N \Rightarrow \quad R = 1/\sqrt{4\pi}$ compactification radius

Operator Mappings

* compactification requires that the observables be invariant under $\phi \to \phi + 2\pi n R$, with $n \in \mathbb{Z}$

$\Rightarrow$ $V_\alpha(\phi) = \exp(i\alpha\phi)$ is allowed if $\alpha = \frac{n}{R} = n\sqrt{4\pi}$

scaling dimension $= n^2$

$$\phi = \phi_R + \phi_L \,, \quad \vartheta = -\phi_R + \phi_L \,, \quad \vartheta(x_0, x_1) = \int_{-\infty}^{x_1} dx_1' \, \Pi(x_0, x_1')$$

$$\Rightarrow \quad \partial_\mu \phi = \varepsilon_{\mu\nu} \partial^\nu \vartheta \qquad \text{(Cauchy-Riemann)}$$

Fermion Operators (Mandelstam, 1975)

$$\psi_R(x) = \frac{1}{\sqrt{2\pi a}} :\exp(i\sqrt{4\pi}\,\phi_R):\,, \quad \psi_L(x) = \frac{1}{\sqrt{2\pi a}} :\exp(-i\sqrt{4\pi}\,\phi_L)$$

$$\bar{\psi}\psi = \psi_R^\dagger \psi_L + \psi_L^\dagger \psi_R \longleftrightarrow \frac{1}{2\pi a} :\cos(\sqrt{4\pi}\,\phi):$$

mass operators: $i\bar{\psi}\gamma_5\psi \equiv i(\psi_R^\dagger \psi_L - \psi_L^\dagger \psi_R) \leftrightarrow \frac{1}{\sqrt{2\pi a}} :\sin(\sqrt{4\pi}\,\phi):$

$\Big\}$ scaling dimension 1

$g = m/(2\pi a)$

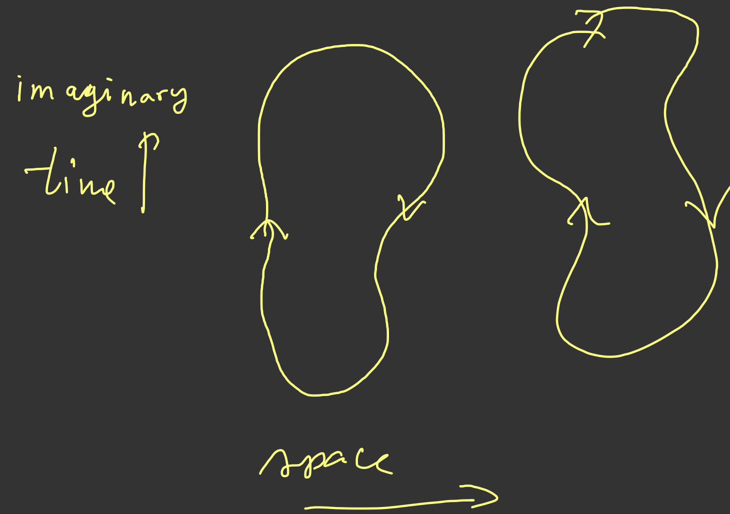
$$\mathcal{L} = \bar{\psi} i \gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi \longleftrightarrow \mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - g\cos(\sqrt{4\pi}\,\phi)$$

Sine-Gordon!

3D: Particle – Vortex duality

(Peskin; Stone & Thomas; Dasgupta, Halperin) (~ 1978 – 1981)

* Global $U(1)$ symmetry
* 3D XY model (superfluid)
* High T: gas of closed loops with short-range interactions (i.e. particle worldlines)
* Low T: closed vortex loops w/ Biot–Savart interactions
* Particle – vortex duality



Simple derivation of 3D Particle-Vortex Duality

* We will follow the same procedure we used in 2D

* $\theta(x)$: phase field of a 3D complex field
* $A_\mu(x)$: background gauge field that will create vortices

$$Z[A_\mu] = \int \mathcal{D}\theta \, \exp\left(-\frac{1}{2g}\int d^3x \, (\partial_\mu \theta - A_\mu)^2\right)$$

vorticity $\quad \omega_\mu(x) = \varepsilon_{\mu\nu\lambda}\partial_\nu A_\lambda \equiv 2\pi \sum_k l^k_\mu(x)\, \delta^3(x - x_k)$

vortex loops

$$\partial_\mu \omega_\mu = 0 \iff \partial_\mu l^k_\mu = 0$$

Hubbard-Stratonovich

$$Z[A] = \int \mathcal{D}\theta \, \mathcal{D}b_\mu \, \exp\left(-\int d^3x \, \frac{g}{2} b_\mu^2 + i\int d^3x \, b_\mu (\partial_\mu \theta - A_\mu)\right)$$

$$\equiv \int \mathcal{D}b_\mu \, \delta(\partial_\mu b_\mu) \exp\left(-\int d^3x \, \frac{g}{2} b_\mu^2 + i\int d^3x \, b_\mu A_\mu\right)$$

$$\partial_\mu b_\mu = 0 \implies b_\mu = \varepsilon_{\mu\nu\lambda}\partial_\nu a_\lambda \; ; \; f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$$

$$\Rightarrow Z[A] = \int \mathcal{D}a_\mu \exp\left(-\int d^3x \, \frac{g}{4} f_{\mu\nu}^2 + i \int d^3x \; a_\mu \underbrace{\varepsilon_{\mu\nu\lambda}\partial_\nu A_\lambda}_{\omega_\mu}\right)$$

Note: These steps contain the statement that in 3D the dual of a Goldstone field (θ) is a gauge field (a_μ)

compactification of $\theta \iff$ charge quantization

Next we sum over vortex configurations

$$Z = \sum_{\{l_\mu^k\}} \delta(\partial_\mu l_\mu^k) \int \mathcal{D}a_\mu \exp\left(-\int d^3x \, \frac{g}{4} f_{\mu\nu}^2 + i \sum_k l_\mu^k(x_k) a_\mu(x_k)\right)$$

upon adding an energy/length to the vortices and a short range repulsion (no crossing)

$$\Rightarrow Z = \int \mathcal{D}a_\mu \mathcal{D}\phi \mathcal{D}\phi^* \exp\left(-\int d^3x \left[\frac{g}{4} f_{\mu\nu}^2 + |D_a\phi|^2 + \underbrace{m^2}_{m^2>0}|\phi|^2 + \lambda|\phi|^4\right)\right.$$

* The theory we derived is the 3D Abelian-Higgs model
  This is the same as a superconductor ϕ coupled to a fluctuating e.m. field a_μ
* Introduce a probe field $B_\mu \Rightarrow A_\mu \to A_\mu + q B_\mu$
  This leads in the dual theory to an extra term

$$\exp\left(-i \int d^3x \, q \, B_\mu \, \varepsilon_{\mu\nu\lambda} \partial_\nu a_\lambda\right) \qquad (q \in \mathbb{Z} \text{ charge})$$

$$\Rightarrow \quad \text{current} \quad j_\mu \longleftrightarrow q \, \varepsilon_{\mu\nu\lambda} \partial_\nu a_\lambda$$

* For $m^2 < 0$ we can run the duality backwards and map the Higgs-Superconducting phase to the unbroken phase of the XY model

Field Theory Picture of Particle-Vortex Duality

* Theory A

background field

$$\mathcal{L} = |D_A \phi|^2 - m^2 |\phi|^2 - u|\phi|^4 \quad , \quad D_A \equiv \partial - iA$$

* Theory B

external field

$$\mathcal{L} = |D_a \varphi|^2 + m^2 |\varphi|^2 - u|\varphi|^4 + \frac{1}{2\pi} \varepsilon_{\mu\nu\lambda} A^\mu \partial^\nu a^\lambda - \frac{1}{4e^2} f_{\mu\nu}^2$$

dynamical field

$$j_\mu \longleftrightarrow \frac{1}{2\pi} \varepsilon_{\mu\nu\lambda} \partial^\nu a^\lambda \qquad \text{(particle-vortex)}$$

Duality maps the unbroken phase of A to the Higgs phase of B
broken phase of A to the unbroken phase of B

* Wilson-Fisher Fixed Points are mapped into each other

Generalization: Web of Dualities

① **Particle-Vortex duality** (Peskin, 1978; Dasgupta & Halperin, 1981)

$$\to |D_A \Phi|^2 - m^2 |\Phi|^2 - u|\Phi|^4 \longleftrightarrow |D_b \varphi|^2 + m^2|\varphi|^2 - u|\varphi|^4 + \frac{1}{2\pi} A\,db + \text{Maxwell}$$

external A ; dynamical b

$$\boxed{J_\mu \longleftrightarrow \frac{1}{2\pi}\varepsilon_{\mu\nu\lambda}\partial^\nu b^\lambda}$$

; vortices $\longleftrightarrow$ particles

② **Bosonization** (Fradkin & Schaposnik, 1994; Seiberg, Senthil, Wang & Witten, 2016)

$$\bar\Psi(i\not{D}_A - M)\Psi - \frac{1}{8\pi} A\,dA \longleftrightarrow |D_a\phi|^2 - m^2|\phi|^2 - u|\phi|^4 + \frac{1}{4\pi} a\,da + \frac{1}{2\pi} a\,dA$$

Dirac fermion $\longleftrightarrow$ monopole ;

$$\boxed{\bar\Psi\gamma^\mu\Psi \longleftrightarrow \frac{1}{2\pi}\varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda}$$

③ **Fermion Particle-Vortex duality** (Son, 2015; Metlitski & Vishwanath, 2016)

$$\bar\Psi(i\not{D}_A - M)\Psi - \frac{1}{8\pi} A\,dA \Leftrightarrow \bar\chi(i\not{D}_a + M)\chi + \frac{1}{8\pi} a\,da - \frac{1}{2\pi} a\,db + \frac{2}{4\pi} b\,db - \frac{1}{2\pi} b\,dA$$

"QED_3"

* In general dimension duality often maps theories with ≠ character and symmetry
* In $D=4$ ⇒ gauge theory ⟺ gauge theory
* There are many other dualities
* AdS / CFT ⟺ gauge / gravity duality
* S and T duality in String Theory
(S duality is related to particle-vortex duality)
* Conjectured web of dualities in 2+1 dimensions
* Fermion ⟺ Boson duality
* Can we "derive" these conjectures?

Strategy for a derivation (Goldman & EF 2018)

* We will use generalized loop models near criticality but still in the gapped phases
* Generalization of the particle-vortex duality
* We consider loop models in 2+1 dimensions
* Assume that the loops cannot intersect
* Include phase factors for linking numbers
* Frame the loops and include self-linking and Berry phase factors $\Rightarrow$ fractional spin

Loop Models in 2+1 Dimensions

$$Z[A] = \sum_{\{l_\mu\}}{}' \delta(\partial_\mu l_\mu)\, e^{-S[l] + i\pi\Phi[l]}$$

background field

loop configurations ("conserved currents")

weight per unit length + interactions

$\Phi[l]$ = linkig number + self-linking number + Berry phase of the frame

Linking # of two loops l_1 and l_2

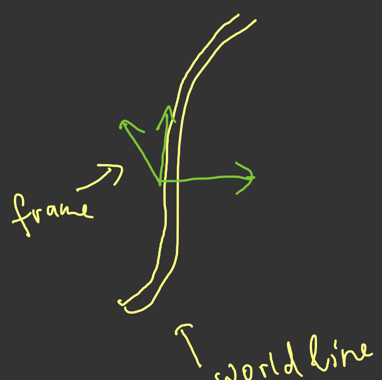
$\Phi = 2 \times$ linking # of l_1 with l_2 +

$+ W[l_1] + W[l_2]$

$W[l] = SL[l] - T[l] \simeq$ "writhe"

self-linking

twist



Twist $\quad T[\ell] = \frac{1}{2\pi} \int_0^1 ds \int_0^1 du \;\; \hat{e} \cdot \partial_s \hat{e} \times \partial_u \hat{e}$

$\hat{e}$ ↑ tangent

$T[\ell]$ in general is <u>not</u> quantized and depends on the metric } Berry phase of the frame

Linking # of ℓ_1 and ℓ_2

$$\left\langle e^{\,i \oint_{\ell_1 \cup \ell_2} dz_\mu \cdot a_\mu} \right\rangle_{\text{Chern Simons}} \equiv e^{\,\frac{i\pi}{k} N_\gamma} \leftarrow \text{topological invariant}$$

N_γ ↑ linking #

wilson loop in Chern-Simons Theory (Witten 89')

$$S = \frac{k}{4\pi} \int d^3x \;\, a\, d\, a$$

$$k \in \mathbb{Z}$$

Example (~ Polyakov 89')

* $Z_{\text{fermion}} = \text{Det}(i\not{\partial} - M) \cong \int \mathcal{D}j_\mu \, \delta(\partial_\mu j^\mu) \, e^{-|M| L[j] - i\pi \, \text{sgn}(M) \, \Phi[j]}$

loop representation

linking #

+ spin factor

$k=1$

* $\mathcal{L}_{\text{boson}} = |D_a \phi|^2 - m^2 |\phi|^2 - u |\phi|^4 + \frac{1}{4\pi} a\, da + \frac{1}{2\pi} a\, dA$

$$Z[A] = \int \mathcal{D}j_\mu \, \mathcal{D}a_\mu \, \delta(\partial_\mu j_\mu) \, e^{-|m| L[j] + i S[j, a, A]}$$

$$S[j, a, A] = \int d^3x \left[j_\mu (a_\mu - A_\mu) + \frac{1}{4\pi} a\, da - \frac{1}{4\pi} A\, dA + \cdots \right]$$

Integrating over $\underline{a_\mu}$ $\Rightarrow$ $-\pi \, \Phi[j] + \int d^3x \left(j_\mu A_\mu - \frac{1}{4\pi} A\, dA \right)$

$$\Rightarrow \quad \mathcal{L}_{\text{fermion}} = \bar{\Psi}\,(i \not{D}_A - M)\,\Psi - \frac{1}{8\pi} A dA \qquad \text{with} \quad M < 0$$

$$\uparrow$$

anomaly (η invariant)

$$Z_{\text{fermion}}[A, M<0]\; \underbrace{e^{-\frac{i}{2} S_{CS}[A]}}_{} = \int \mathcal{D}j\, \delta(\partial \cdot j)\, e^{-|M| L[j]}\, e^{i S_{\text{fermion}}[j, A, M<0]}\, e^{-\frac{0}{2} S_{CS}[A]}$$

$$S_{\text{fermion}}[j, A, M<0] = \int d^3x \left[j \cdot A - \frac{1}{8\pi} A dA \right] - \pi\, \bar{\Phi}[j]$$

* To get the bosonization identity for $M > 0$ one uses bosonic particle – vortex duality

* In the fermionic theory $M < 0 \Leftrightarrow M > 0 \Rightarrow \sigma_{xy} = 0 \Longleftrightarrow \frac{e^2}{h}$

* In the bosonic theory this is the transition from broken to the unbroken phase

Fermion Particle-Vortex Duality

* Duality from a free Dirac fermion $\longleftrightarrow$ QED_3 with a quantized CS term

$$\bar{\Psi}(i\not{D}_A + M)\Psi - \frac{1}{8\pi} AdA \longleftrightarrow \bar{\chi}(i\not{D}_a - M)\chi + \frac{1}{8\pi} ada - \frac{1}{2\pi} adb + \frac{2}{4\pi} bdb - \frac{1}{2\pi} bdA$$

$\downarrow$ (left) $\qquad\qquad \downarrow$ (right)

loop model

$$\rightarrow \int d^3x \; j_\mu A^\mu + \pi \Phi[j] \longleftarrow -\pi \Phi[j] + \int d^3x \left[j.a - \frac{1}{2\pi} adb + \frac{2}{4\pi} bdb - \frac{1}{2\pi} bdA \right]$$

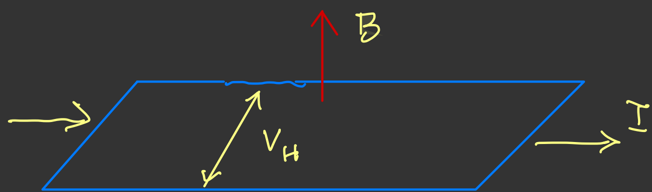
integrate out b_μ and a_μ

$$Z_{fermion}[A, M] = Z_{QED_3}[A, -M] \; ; \; Z_f[A, -M] = Z_{QED_3}[A, M]$$

Currents: $\bar{\Psi}\gamma^\mu\Psi \longleftrightarrow \frac{1}{2\pi}\varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda$

Application: Fractional Quantum Hall States

In the beginning .... two-dimensional electron gases in large magnetic field



$\sigma_{xy} = \nu \dfrac{e^2}{h}$, $\sigma_{xx} \to 0$ $(T \to 0)$

no dissipation

Laughlin: $\Psi_m(z_1, \dots, z_N) \sim \prod_{i<j} (z_i - z_j)^m \; e^{-\frac{1}{4 l_0^2} \sum_{j=1}^{N} |z_j|^2}$ (1983)

filling fraction $\nu = \frac{1}{m}$; $\{z_j\}$: electron coordinates $(z = x + iy)$

l_0: magnetic length

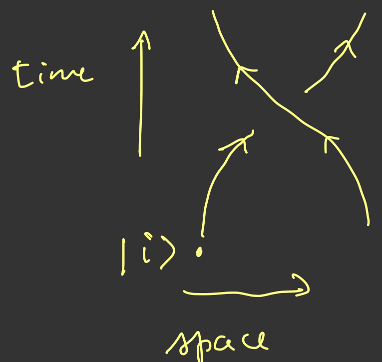
Jain: composite fermion : electron + $(m-1)$ fluxes (m odd)

FQH state : IQH state of composite fermions

$\to \nu_\pm(p, s) = \dfrac{p}{2sp \pm 1}$ $p = 1, 2, \dots$ $s = 0, 1, 2 \dots$ (Laughlin : $p = 1$, $+$) $m = 2s + 1$

odd denominators

* The excitations of FQH fluids are vortices ("quasiholes") that
(a) carry fractional charge $q = \frac{1}{2sp \pm 1}$ ←
(b) fractional braiding statistics (anyons) (Halperin '84, Arovas, Schrieffer and Wilczek '84)
(c) m degenerate ground states on a torus (topological protection)



amplitude $\sim e^{i\varphi}$

✓ $\varphi = \frac{\pi}{2sp \pm 1}$ ⇒ anyons labelled by one-dimensional representations of the Braid Group

✓ $e^{i\varphi_1} e^{i\varphi_2} \rightarrow e^{i(\varphi_1 + \varphi_2)}$ ("fusion")

Wen: ~1990 Effective Field Theory (Laughlin states)

$$\mathcal{L} = \frac{m}{4\pi} \varepsilon_{\mu\nu\lambda} a^\mu \partial^\nu a^\lambda + \underbrace{\frac{e}{2\pi} \varepsilon_{\mu\nu\lambda} \partial^\nu a^\lambda}_{\text{current}} A^\mu + j_\mu a^\mu \equiv$$

Chern-Simons term
a_μ: hydrodynamic gauge field

e.m. field

vortex world lines

Duality at the FQH Plateau Transition

(Hart Goldman & EF, 2019)

* Limiting value of the Jain sequence

$$\lim_{p\to\infty} \frac{p}{2np \pm 1} = \frac{1}{2n}$$

* In this limit the average CS field cancels A_μ
* Halperin-Lee-Read: this is a "Fermi Liquid"
* Good phenomenology but...
* singular forward scattering interactions and violation of particle-hole symmetry at $\nu = 1/2$ $(\nu \leftrightarrow 1-\nu)$

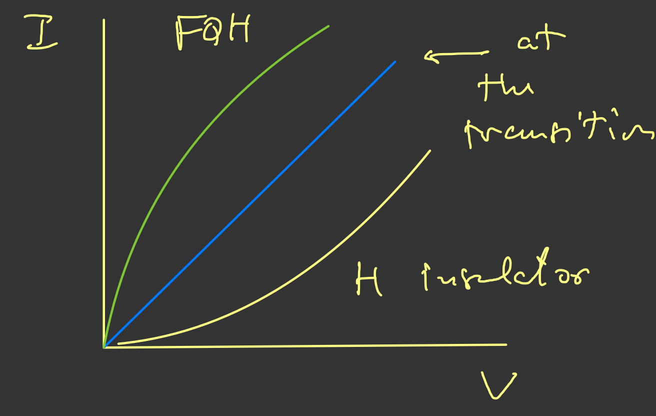
Symmetry of the I–V curves at the transition

* The I–V curves show a "mirror" symmetry at all transitions

* For general Jain states

$$\nu = \frac{p}{2np+1} \leftrightarrow \nu' = \frac{1+p}{2n(1+p)-1}$$

* For $\nu = 1/2 \Longleftrightarrow$ PH symmetry



I–V curves at the $0 \leftrightarrow \frac{1}{3}$ transition
$\left(\nu \cong \frac{1}{4} \right)$
Hall insulator $\leftrightarrow$ FQH

$\nu = \frac{1}{2}$: Son's Conjecture

General case : $\nu = \frac{1}{2n}$

$$\mathcal{L}_{1/2n} = i\bar{\Psi} \not{D}_a \Psi - \frac{1}{4\pi}\left(\frac{1}{2} - \frac{1}{2n}\right) a\,da + \frac{1}{2\pi}\frac{1}{2n} a\,dA + \frac{1}{2n}\frac{1}{4\pi} A\,dA$$

$\partial \times A = B$

a : flux attachment

Electron filling $\quad \nu = \frac{2\pi}{B}\left\langle \frac{\delta \mathcal{L}_{\nu=1/2n}}{\delta A_0} \right\rangle = \frac{1}{2n}\left(1 + \frac{b_*}{B}\right)$

$$b^* = \partial \wedge a = 0 \Rightarrow \nu = \frac{1}{2n}$$

Composite fermion ψ Fermi Surface set by a_0

$$\rho_\psi = \frac{1}{2\pi}\left(\frac{1}{2} - \frac{1}{2n}\right) b_* - \frac{1}{2\pi}\frac{B}{2n}$$

$$\Rightarrow \quad \nu_\psi = 2\pi \frac{\rho_\psi}{b_*} = \frac{1}{2} + \frac{\nu}{1-2n\nu}$$

filling fraction of ψ

$\Rightarrow$ If $\nu_\psi = p + \frac{1}{2}$ $\Rightarrow$ $\nu = \dfrac{p}{2np+1}$

(Dirac)

But, if $\nu_\psi \to -\nu_\psi$ $\Rightarrow$ $\nu = \dfrac{p}{2np+1} \to \dfrac{1+p}{2n(1+p)-1}$ ✓ !

(PH transf.)

$\Rightarrow$ PH transf. of the Dirac composite fermion is equivalent to the reflection symmetry !

Self-Duality at the Transition

* Use fermion - boson duality

$$\mathcal{L}_{1/2n} \longleftrightarrow |D_{g-A}\phi|^2 - |\phi|^4 + \frac{1}{4\pi}\frac{1}{2n-1} g\,dg \quad \leftarrow \quad \nu_\phi$$

* Followed by a (boson) particle - vortex duality

$$\mathcal{L}_{1/2n} \longleftrightarrow |D_h \varphi|^2 - |\varphi|^4 - \frac{2n-1}{4\pi} h\,dh + \frac{1}{2\pi} h\,dA$$

$$-\frac{1}{\nu_\varphi}$$

* $\nu = \frac{1}{2n} \Longleftrightarrow \nu_\phi = -\nu_\varphi = 1$

* Reflection symmetry $\nu_\phi(\nu) = -\nu_\varphi(\nu')$

* Reflection $\Longleftrightarrow$ boson - vortex symmetry !

* Reflection symmetry <u>at</u> $\nu = \frac{1}{2n} \Longleftrightarrow$ boson self-duality !

Non-Abelian States: Moore-Read (1991)

$$\Psi_{MR}(\{z_i\}) \sim \underbrace{\mathrm{Pf}\left(\frac{1}{z_i - z_j}\right)}_{\text{Pfaffian}} \underbrace{\prod_{i<j} (z_i - z_j)^n}_{} \; e^{-\frac{1}{4\ell_0^2}\sum_{i=1}^{N} |z_i|^2}$$

Pfaffian: expectation value of chiral Majorana fermions $\quad \chi(z) = \chi^\dagger(z)$

Propagator: $\langle \chi(z)\, \chi(w) \rangle = \dfrac{1}{z-w}$

$$\mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) = \langle \chi(z_1) \cdots \chi(z_N) \rangle$$ ← "paired states" ($p_x + i p_y$ superconductor)

$\varphi(z)$: chiral boson $\qquad \varphi(z) \sim \varphi(z) + 2\pi\sqrt{n}$

$$\Psi_{MR} \sim \langle \chi(z_1) \cdots \chi(z_N) \rangle \left\langle \left(\prod_{i=1}^{N} e^{i\sqrt{n}\,\varphi(z_i)} \right) e^{-\int d^2 z' \sqrt{n}\, \rho_0\, \varphi(z')} \right\rangle$$

Filling fraction: $\nu = \frac{1}{n}$

n even → fermions ; n odd ↔ bosons ; e.g. $\nu = \frac{1}{2}$ fermions

$\nu = 1$ bosons

Generalization: Read-Rezayi states (RR) (1998)

Based on $\mathbb{Z}_k$ parafermions (and $SU(2)_k$)

$$\psi_n(z) * \psi_m(z') \sim \frac{1}{(z-z')^{\Delta_n+\Delta_m-\Delta_{n+m}}} \psi_{n,m}(z') + \dots$$

Fradkin & Kadanoff (1980) (!)

$$\Delta_n = \frac{n(k-n)}{k}, \quad n,m = 1,\dots,k-1$$

RR states use the Parafermion CFT (Zamolodchikov & Fateev, 1985
Gepner & Qiu, 1987)

$$\Psi_{RR}(\{z_i\}) \sim \langle \psi_1(z_1)\dots\psi_1(z_N)\rangle \prod_{i<j}(z_i-z_j)^{M+\frac{2}{k}} \times \text{gaussians}$$

vanishes when $k+1$ particles come together.
clustering

$M \in \mathbb{Z}$ divisible by k; M even: bosons, M odd: fermions; $\nu = \frac{k}{Mk+2}$

The most interesting case is $k=3$ ($\mathbb{Z}_3$) ($\nu = \frac{3}{2}$ (B), $\frac{3}{5}$ (F))

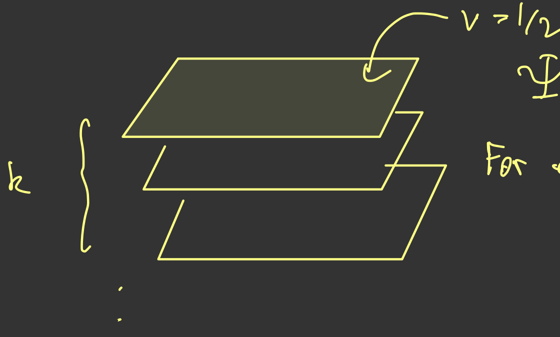
In addition to the $\mathbb{Z}_3$ parafermion, it has a Fibonacci anyon τ

Fusion rule: $\tau * \tau = I + \tau$ $\Rightarrow$ its unitary braiding matrices cover $SU(2)$
(Fibonacci sequence) $\Rightarrow$ universal quantum computer

Effective Field Theory Approaches (Fradkin, Nayak, Schoutens, 1999) (Goldman, Sohal, EF, 2019, 2020)

We will discuss bosons for simplicity, $\nu = \frac{k}{2}$

Consider k layers of bosons in a $\nu = \frac{1}{2}$ Laughlin state



$$\Psi_{1/2} \sim \prod_{i<j} (z_i - z_j)^2 \times \text{gaussians}$$

For each layer $\mathcal{L} = \frac{2}{4\pi} \epsilon_{\mu\nu\lambda} a^\mu \partial^\nu a^\lambda + \dots$ ⟵ $U(1)_2$

$$\equiv \frac{2}{4\pi} a\,da + \frac{1}{2\pi} A\,da + \dots$$

Symmetry $\underbrace{U(1)_2 \times \cdots \times U(1)_2}_{k \text{ factors}}$

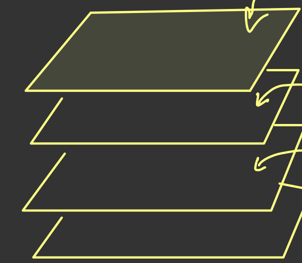
Chern-Simons level-rank duality $U(1)_2 \longleftrightarrow SU(2)_1$ group is non-abelian

$I,\ e^{i\varphi/\sqrt{2}}$ $\qquad j = 0, \frac{1}{2}$ $\qquad$ the braids are abelian

Q: how to get to a state with non-abelian statistics?

Hint: somehow we need a theory on $SU(2)_k$

you need $U(1)_2 \times \cdots \times U(1)_2 \longrightarrow SU(2)_k$



$\langle \phi_j \, \phi_{j+1} \rangle \neq 0$

(A) ① use the Chern-Simons level-rank duality

$SU(2)_1 \times \cdots \times SU(2)_1$

② construct a condensate $\rightarrow SU(2)_k$

The 1999 paper did this by condensing pairs of excitations on two layers at a time

$\Rightarrow$ Higgs (Meisner) mechanism projects onto a state with symmetry $SU(2)_k$ (clustering)

1999 was basically right (but not completely)

$\Rightarrow$ Dualities solve the problem

Construction of a Fibonacci FQH state (Goldman, Sohal, EF, 2021)

* Want a FQH state with only Fibonacci anyons

$\tau * \tau = 1 + \tau$ (and no other anyons)

$\Rightarrow$ universal quantum computing (3 τ's form a qubit)

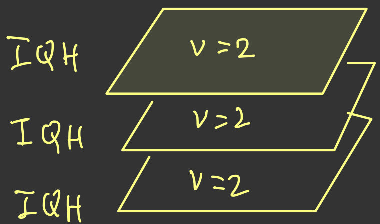
* Topological QFT?

$$(G_2)_1 \leftrightarrow U(2)_{3,1} = \frac{SU(2)_3 \times U(1)_2}{\mathbb{Z}_2}$$

$$\mathcal{L}_{Fib} = \frac{3}{4\pi} \mathrm{Tr}\left[\underset{\substack{\uparrow \\ SU(2)\text{ gauge field}}}{a}\, d a - \frac{2}{3} i a^3\right] - \frac{1}{4\pi} \mathrm{Tr}[a]\, d\, \underset{\substack{\uparrow \\ U(1)_2}}{\mathrm{Tr}[a]} + \frac{1}{4\pi} \underset{\substack{\uparrow \\ \text{background}}}{A}\, d\, \mathrm{Tr}[a]$$

$\Rightarrow \quad \nu = 2 \quad \left(\sigma_{xy} = 2 \frac{e^2}{h} \right)$

* Start with 3 layers of Diracs at $\nu = 2 \to 1$ transition (IQH)



$$\mathcal{L} = \sum_{n=1}^{3} \left[\bar{\Psi}_n (i \not{D}_A - M) \Psi_n - \frac{3}{2} \frac{1}{4\pi} A\, dA \right]$$

$$D_A = \partial - iA$$

Parity anomaly

Duality: Free Dirac $\Psi \longleftrightarrow$ Wilson–Fisher boson ϕ + $U(N)_1$

OK since $U(N)_1 \longleftrightarrow \mathcal{L}_{eff} = -\frac{N}{4\pi} A\, dA$ (trivial)

* Set $N=2$

$$\mathcal{L} = \sum_n \left[|D_{a_n} \phi_n|^2 - r|\phi_n|^2 - |\phi|^4 + \mathcal{L}_{CS}[a_n] \right] + \frac{1}{2\pi} A\, d\, \mathrm{Tr}[a_1 - a_2 + a_3]$$

* Clustering: $\langle \Gamma_{mn} \rangle = \langle \phi_m^\dagger \phi_n \rangle \neq 0 \ (m \neq n)$, $\langle \phi_n \rangle = 0$

$\Rightarrow$ pins $a_1 = a_2 = a_3 \equiv a \Rightarrow \frac{1}{2\pi} A\, d\, \mathrm{Tr}[a_1 - a_2 + a_3] \equiv \frac{1}{2\pi} A\, d\, \mathrm{Tr}[a]$

* The physical densities are pinned $\rho_1 = -\rho_2 = \rho_3$
  $\Rightarrow$ layer exchange symmetry is broken

$$\Rightarrow \mathcal{L}_{U(2)_3} = 3\,\mathcal{L}_{CS}[a] + \frac{1}{2\pi} A\, d\, \mathrm{Tr}[a]$$

* To get Fibonacci $\Leftrightarrow$ attach a unit of flux to the fermions
  $\Rightarrow$ fermions $\rightarrow$ bosons

flux attachment: $3\,\mathcal{L}_{CS}[a] + \frac{1}{2\pi} b\, d\, \mathrm{Tr}[a] + \frac{1}{4\pi}(b+A)\, d\,(b+A)$

fluctuating
U(1) gauge field

* Integrating out $b_\mu \Rightarrow$ obtain $\mathcal{L}_{Fib}$!
  $\Rightarrow$ interpret $\underline{\phi^\dagger t^a \phi}$ as the Fibonacci anyon τ

* Alternatively we can attach (+) flux to layers 1, 3 and (−) to layer 2

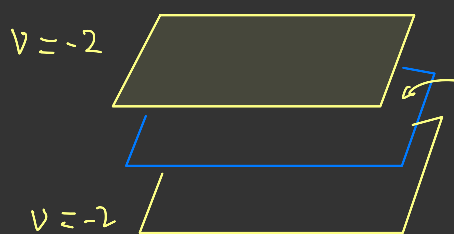
<u>before</u> clustering

$\Rightarrow$ layers 1, 3 become $|D_A \Phi|^2 + \frac{2}{4\pi} A dA$ (trivial)

layer 2 : $|D_\alpha \Phi|^2 + \frac{2}{4\pi}\alpha d\alpha + \frac{2}{4\pi}\beta d\beta + \frac{1}{2\pi}\alpha d\beta + \frac{1}{2\pi}\beta dA$

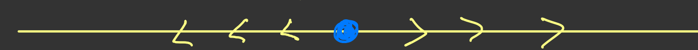
layer 2 $\Rightarrow$ Halperin (2,2,1) state

Transition (2,2,1) $\leftrightarrow$ Fibonacci



(2,2,1) $\Rightarrow$

Fibonacci $U(2)_{3,1}$ (2,2,1) Halperin



$\langle \Gamma_{mn} \rangle \neq 0$ $\langle \Gamma_{mn} \rangle = 0$

One can use this construction to derive the Fibonacci wave function!

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* Non-Abelian dualities can be used to understand the landscape of non-abelian FQH states

* define physical parent state

* construct ideal wave functions using CFT methods

* hopefully to find simple Hamiltonians!

* Opens a window to universal TQC!

References: Goldman, Sohal, EF PRB 100, 115111 (2019)
102, 195151 (2020)
103, 235118 (2021)