Holographic quantum matter

1. Introduction to AdS/CFT

Andrew Lucas

University of Colorado Boulder

Pontifical Catholic University of Chile

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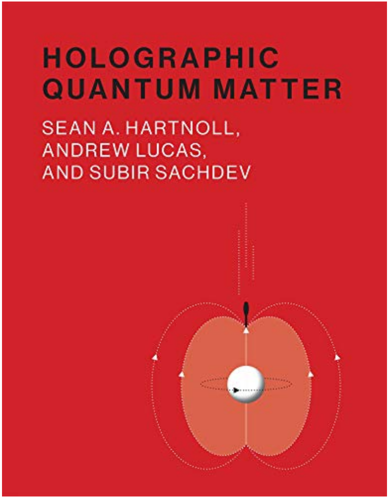

arXiv:1612.07324

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duality = a (non-trivial) equivalence between the two descriptions

[Maldacena; *Int. J. Theor. Phys.* **38** 1113 (1999)]

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- quantum phases of matter without quasiparticles **Lecture 2**
- where real-time response is "easy" to calculuate **Lecture 3**
- and models have Planckian dynamics: **Lecture 4**

$$\tau_{\mathrm{eq}} \sim \frac{\hbar}{k_{\mathrm{B}}\, T}.$$

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At critical value of t/U, *universal* physics captured by continuum quantum field theory! O(2) Wilson-Fisher fixed point:

[Fisher++; *Phys. Rev.* **B40** 546 (1989)]

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Access perturbatively?! Use vector field with $N \to \infty$ components:

$$\mathcal{L} = -\partial_\mu \bar{\phi}^a \partial^\mu \phi^a - s_\mathrm{c} \phi^a \phi^a - u \left(\phi^a \phi^a \right)^2$$

Work in $d = 3 - \epsilon$ **spatial** dimensions? Take $N \to 2$?

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$$\sigma(\omega) \sim \frac{1}{\mathrm{i}\omega}\left(G^{\rm R}_{J_xJ_x}(\omega) - G^{\rm R}_{J_xJ_x}(0)\right) \neq \frac{c_1\,T}{\frac{1}{N}\,c_2\,T - \mathrm{i}\omega} + \text{small}.$$

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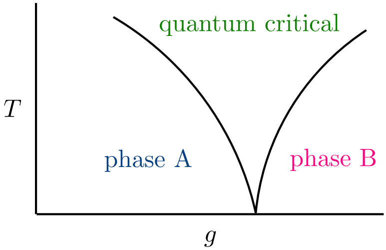
The Drude peak is an artifact of quasiparticles at large N. This is *qualitatively wrong* in $N = 2$ quantum critical regime:



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The reason we do better is that

holography assumes important degrees of freedom are $T^{\mu\nu}$, J^{μ}, etc.

The pathological $N \to \infty$ limit underlying holography shows up elsewhere, and should be accounted for.

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CFTs are very powerful: 2- and 3-point correlators almost unique!

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle = \frac{C_{\mathcal{O}\mathcal{O}}}{|x-y|^{2\Delta}},$$
$$\langle \mathcal{O}(x)\mathcal{O}(y)\mathcal{O}(z)\rangle = \frac{C_{\mathcal{O}\mathcal{O}\mathcal{O}}}{|x-y|^{\Delta}|x-z|^{\Delta}|y-z|^{\Delta}}.$$

Δ is the **scaling dimension**.

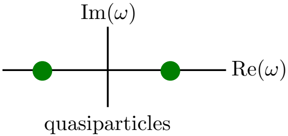
$C_{\mathcal{O}\mathcal{O}}$ are coefficients of **operator product expansion**:

$$\mathcal{O}(x)\mathcal{O}(0) = \frac{C_{\mathcal{O}\mathcal{O}}}{|x|^{2\Delta}} + \frac{C_{\mathcal{O}\mathcal{O}\mathcal{O}}}{C_{\mathcal{O}\mathcal{O}}}\frac{\mathcal{O}(0)}{|x|^{\Delta}} + \cdots \quad (|x| \to 0)$$

A CFT with quasiparticles: free boson in d spatial dimensions:

$$\langle \varphi(x)\varphi(0)\rangle = \frac{1}{|x|^{d-1}},$$

$$\langle \varphi(k)\varphi(-k)\rangle = \frac{1}{k^2}.$$



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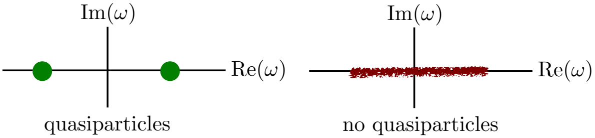

A CFT without quasiparticles: generic Δ:

$$\langle \mathcal{O}(k)\mathcal{O}(-k)\rangle = |k|^{2\Delta-d-1}.$$

AdS space

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(Semi)**classical** gravity in asymptotically anti-de Sitter space!

Gravity is a theory of **curved spacetime** where the distance between x^μ and $x^\mu + \mathrm{d}x^\mu$ is

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu.$$

The classical (Einstein) theory of gravity is

$$S = \int \mathrm{d}^D x \sqrt{-g}\left(\frac{R}{16\pi G} + \mathcal{L}_{\mathrm{matter}}\right)$$

where $g = \det(g_{\mu\nu})$ and R is the Ricci scalar:

$$\text{"}R \sim \nabla^2 g_{\mu\nu}\text{"}$$

Anti-de Sitter (AdS) space is a solution to the D-dimensional Euler-Lagrange equations with

$$\mathcal{L} = \frac{1}{16\pi G}\left[R + \underbrace{\frac{D(D-1)}{L^2}}_{\text{negative cosmological constant}}\right].$$

In a useful (Poincaré) coordinate patch, the metric is:

$$\mathrm{d}s^2 = \frac{L^2}{r^2}\left[\underbrace{\mathrm{d}r^2}_{\text{bulk}} + \underbrace{\mathrm{d}\mathbf{x}_d^2 - \mathrm{d}t^2}_{d\text{ space + time}}\right].$$

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Light from the bulk ($r > 0$) of AdS can hit the boundary ($r = 0$ in finite time, so boundary conditions will be important!

$$\mathrm{d}s^2 = 0 = \frac{L^2}{r^2}\left[\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 - \left(\frac{\mathrm{d}t}{\mathrm{d}t}\right)^2\right] \implies r = r_0 - t.$$

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For AdS_{d+2}:

$$\mathrm{d}s^2 = \frac{L^2}{r^2}\left[\underbrace{\mathrm{d}r^2}_{\text{bulk}} + \underbrace{\mathrm{d}\mathbf{x}_d^2 - \mathrm{d}t^2}_{d\ \text{space + time}}\right],$$

we have:

- translation: $x^\mu \to x^\mu + c^\mu$
- Lorentz transformations on (t, x_i).
- scale invariance: $r \to \lambda r$, $t \to \lambda t$, $x \to \lambda x$
- special conformal transformations

which is also the conformal symmetry group... 🤔

Holographic dictionary

Conjecture: the generating function of $(d+1)$-dimensional CFT is a quantum gravity partition function on AdS_{d+2}:

[Gubser, Klebanov, Polyakov; *Phys. Lett.* **B428** 105 (1998)]

[Witten; *Adv. Theor. Math. Phys.* **2** 253 (1998)]

$$\left\langle \exp\left[\int \mathrm{d}^{d+1}x\; \phi_0(x)\mathcal{O}(x)\right]\right\rangle = Z_{\mathrm{grav}}\left[\phi_{\mathrm{bulk}}(x, r\to 0)\to r^{\#}\phi_0(x)\right].$$

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Approximate Z_{grav} by **semiclassical saddle point** if: 😃

$$G \sim \frac{1}{N^2} \to 0$$

$$L \sim \lambda^{1/4} L_{\mathrm{Planck}} \to \infty$$

Such a limit is reasonable (in some string theories). N is rank of gauge group, and λ is string coupling constant.

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scalar $\mathcal{O}$ of dim. Δ	scalar ϕ with $m^2 = \Delta(\Delta - d - 1)/L^2$
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$$Z[\delta\phi, \delta A_\mu, \delta g_{\mu\nu}] = \left\langle \exp\left[\int \mathrm{d}^{d+1}x \left(\mathcal{O}\delta\phi + J^\mu \delta A_\mu + \frac{1}{2} T^{\mu\nu}\delta g_{\mu\nu}\right)\right]\right\rangle_{\text{CFT}}.$$

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Bulk fields needed for **low dimension** Δ operators. Bulk theory is gravitational since the stress tensor exists and is low dimension!

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Note that $m^2<0$ possible for real/allowed Δ! Plug in to bulk action:

$$S_{\mathrm{bulk}}=\int_\epsilon^\infty \mathrm{d}r\left[-\frac{1}{2}|\nabla\phi|^2-\frac{1}{2}m^2|\phi|^2\right]=\epsilon^{d+1-2\Delta}+k^{2\Delta-d-1}\epsilon^0+\cdots$$

A local counterterm cancels divergences. What remains is:

$$|k|^{2\Delta-d-1}\sim\int\mathrm{d}^{d+1}x\mathrm{e}^{-\mathrm{i}k_\mu x^\mu}\frac{1}{(x_\mu x^\mu)^\Delta}=\langle\mathcal{O}(k)\mathcal{O}(-k)\rangle_{\mathrm{CFT}}$$

The general principle is as follows: (take $L = 1$ for simplicity)

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(For some values of Δ, there exists alternate quantization.)

Holographic dictionary

The bulk action encodes OPE coefficients:

e.g. [Lucas, Sierens, Witczak-Krempa; *JHEP* **07** 149 (**2017**)]

$$S_{\mathrm{bulk}} = \int \mathrm{d}^{d+2}x\sqrt{-g}\left(R - 2\Lambda - C_{\mathcal{O}\mathcal{O}}\left[(\nabla\phi)^2 + m^2\phi^2\right] - C_{\mathcal{O}\mathcal{O}\mathcal{O}}\phi^3 + \cdots\right).$$

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A non-trivial part of this OPE is *always* with stress tensor:

$$\sqrt{-g}\phi^2 \to \sqrt{-g}\left(1 - \frac{1}{2}g^{ab}\delta g_{ab}\right)\delta\phi^2 \subset C_{\mathcal{O}\mathcal{O}T}\delta g_{ab}\delta\phi^2$$

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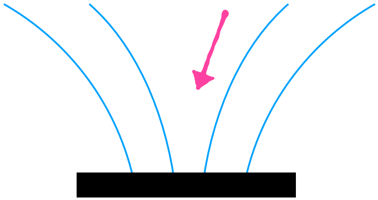
Usually symmetry is broken by $r = 0$ boundary conditions: **Lecture 2**

There is a crucial exception: a **black hole**!



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Most generic x^μ-homogeneous ansatz:

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The most general solution with $g(0) = f(0) = 1$ [AdS bound. cond.]:

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Black hole horizon where $g_{tt} = 0$, at $r = r_0$.

Black holes have a (Hawking) **temperature** T:

e.g. [Bardeen, Carter, Hawking; *Comm. Math. Phys.* **31** 161 (1973)]

$$t \to \mathrm{i}\tau = \mathrm{i}\frac{2}{|f'(r_0)|}\theta, \quad r = r_0 - \frac{r_0^2}{4}|f'(r_0)|\rho^2$$

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Hawking temperature = *boundary theory's temperature.*

Finite temperature: black holes

For gravity, the bulk-boundary correspondence:

$$g_{tt}(r \to 0) = \underbrace{-\frac{1}{r^2}}_{\text{source}} + \underbrace{\frac{r^{d-1}}{r_0^{d+1}}}_{\text{response}}$$

implies that

$$\text{energy density} = \langle T^{tt} \rangle \sim \frac{1}{r_0^{d+1}} \sim T^{d+1},$$

which matches dimensional analysis for a CFT!

Response functions in a black hole background gives us **real-time, thermal correlators**! [Son, Starinets; *JHEP* **09** 042 (**2002**)]

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Boundary thermalization: fall into/grow bulk black hole!

Non-stringy justification for holography: **bulk geometry encodes entanglement structure of dual theory**.

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Review: given qubits on lattice, ground state

$$|\psi_0\rangle = \sum c_{s_1 s_2 s_3 s_4 \cdots} |s_1\rangle |s_2\rangle |s_3\rangle |s_4\rangle \cdots$$

trace out complement of A to get **reduced density matrix** on A:

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Entanglement entropy of A with region A^c:

$$S_A = \mathrm{tr}\left[-\rho_A \log \rho_A\right] \geq 0.$$

Entanglement entropy

In $1+1$-dimensional CFT, entanglement of domain of size ℓ:

$$S_\ell = \frac{c}{6} \log \frac{\ell}{\epsilon}.$$

$\epsilon =$ UV cutoff (lattice size). [Calabrese, Cardy; *J. Stat. Mech.* P06002 (**2004**)]

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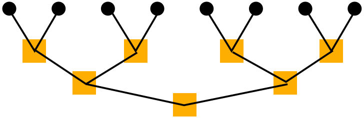
Related to the entanglement structure of CFT ground state (MERA):

[Swingle; *Phys. Rev.* **D86** 065007 (2012)]

$$|\psi_0\rangle = \sum_{a_0,a_1,\cdots,b_0,\cdots} C^{c_0}_{b_0 b_4}\cdots C^{b_0}_{a_0 a_2}\cdots\cdot C^{a_0}{}_{s_0 s_1} C^{a_1}{}_{s_1 s_2}|s_0 s_1 s_2\cdots\rangle$$



Entanglement entropy

In AdS/CFT, in semiclassical limit:

[Ryu, Takayanagi; *Phys. Rev. Lett.* **96** 181602 (2006)]

$$S_A = \frac{1}{4G} \min \text{Area}(\text{bulk surface with boundary } A)$$

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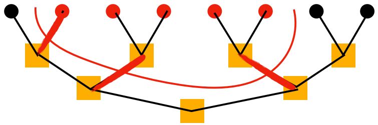
Intuitively – the number of bonds broken in MERA!



Example: 1+1d CFT, interval of length ℓ:

$$\text{Area} = \int \mathrm{d}s = \int \sqrt{\frac{\mathrm{d}r^2}{r^2} + \frac{\mathrm{d}x^2}{r^2}} = \int \frac{\mathrm{d}x}{r} \sqrt{1 + \left(\frac{\mathrm{d}r}{\mathrm{d}x}\right)^2}.$$

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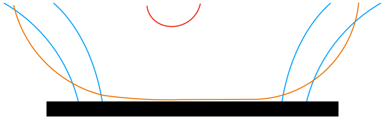
Require $r(\pm\frac{\ell}{2}) = 0$. Euler-Lagrange equation solved by

$$r(x) = \sqrt{\left(\frac{\ell}{2}\right)^2 - x^2},$$

leading to

$$\text{Area} = 2\int_0^{\ell/2} \frac{\frac{\ell}{2}\mathrm{d}r}{(\frac{\ell}{2})^2 - r^2} \to \frac{c}{6}\log\frac{\ell}{\epsilon}.$$

Add a finite temperature:



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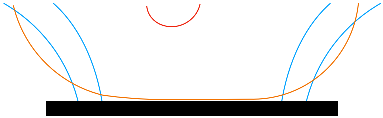

Schematically we find:

$$S_\ell \sim \begin{cases} \frac{c}{6}\log\frac{\ell}{\epsilon} & \ell \ll r_0 \\ \ell/r_0 & \ell \gg r_0 \end{cases} \sim \begin{cases} \frac{c}{6}\log\frac{\ell}{\epsilon} & \ell \ll T^{-1} \\ T\ell & \ell \gg T^{-1} \end{cases} .$$

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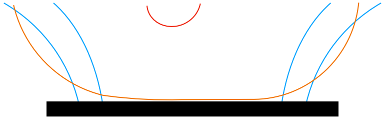

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For large distances we reproduce thermal entropy density:

$$s_{\rm th} \sim \sqrt{g|_{r,t}}$$

or

$$\text{area of horizon} = \text{thermal entropy}$$