Connecting null infinity with spatial infinity

Marc Henneaux

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Penrose conformal compactification of Minkowski space

The Minkowski metric reads, in spherical coordinates,

$$d\Sigma^2 = -dt^2 + dr^2 + r^2 d\Omega^2 \tag{1.1}$$

with

$$d\Omega^2 = d\theta^2 + \sin^2\theta \, d\varphi^2 \tag{1.2}$$

$(-\infty < t < \infty, 0 \leq r < \infty)$. Each point in the (t, r)-plane with $r > 0$ is a 2-sphere of radius r. The Penrose diagram is obtained by first defining the null coordinates

$$u = t - r, \qquad v = t + r, \qquad v \geq u \tag{1.3}$$

in terms of which the metric takes the form

$$d\Sigma^2 = -du dv + \frac{1}{4}(v - u)^2 \, d\Omega^2 \tag{1.4}$$

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We now redefine u and v so as to bring infinity to finite coordinate values. This is done as follows,

$$u = \tan p, \qquad v = \tan q, \qquad -\frac{\pi}{2} < p \leq q < \frac{\pi}{2} \tag{1.5}$$

so that

$$d\Sigma^2 = \frac{1}{4}\frac{1}{\cos^2 p}\frac{1}{\cos^2 q}\left[-4dpdq + \sin^2(q-p)\, d\Omega^2\right], \tag{1.6}$$

and then introduce

$$T = q + p, \qquad R = q - p, \tag{1.7}$$

with $-\pi < T - R \leq T + R < \pi$, i.e., $|T| < \pi, \quad 0 \leq R, \quad R < \pi - T, \quad R < \pi + T$

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This yields

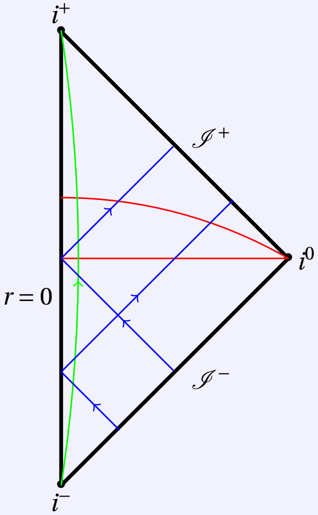
$$d\Sigma^2 = \Psi^{-2} d\bar{\Sigma}^2, \qquad \Psi^2 = 4\cos^2 p \cos^2 q \tag{1.8}$$

with

$$d\bar{\Sigma}^2 = -dT^2 + dR^2 + \sin^2 R \, d\Omega^2 \tag{1.9}$$

While the original Minkowski metric $d\Sigma^2$ blows up at the boundaries, the conformally rescaled metric $d\bar{\Sigma}^2$ is regular there and can be extended to $T - R = -\pi$ and $T + R = \pi$.
This spacetime with the boundary included is called the conformal completion of Minkowski space.
It is depicted by a “Penrose diagram”, in which the angular variables are traditionnally suppressed. Light rays are at 45 degrees since the (t, r) and (T, R) coordinates are related by a conformal transformation. Causality relations are therefore easy to determine.

Penrose diagram of Minkowski space



Penrose diagram of Minkowski space

The boundary can be subdivided into the following regions :

- i^0, called "spatial infinity" is defined by $T = 0$, $R = \pi$; it is a single point (South Pole at $T = 0$).
- i^+, called "future timelike infinity" is defined by $T = \pi$, $R = 0$; it is also a single point.
- i^-, called "past timelike infinity" is defined by $T = -\pi$, $R = 0$; it is again a single point.
- $\mathscr{I}^+$, called "future null infinity" is defined by $T = \pi - R$, with $0 < R < \pi$; it is a cylinder (topology $\mathbb{R} \times S^2$).
- $\mathscr{I}^-$, called "past null infinity" is defined by $T = -\pi + R$, with $0 < R < \pi$; it is also a cylinder.

Penrose diagram of Minkowski space

The reasons for the above terminology are as follows : (i) All spacelike geodesics end at i^0. (ii) All timelike geodesics start at i^- and end at i^+. (iii) All null geodesics start at $\mathscr{I}^-$ and end at $\mathscr{I}^+$. In our case, the geodesics are of course just straight lines and the above assertions are easily verified by going through the successive coordinate transformations leading from (t, r) to (T, R).

The spacelike hyperplanes all reach i_0 and are clearly Cauchy hypersurfaces. For instance, the hyperplane $t = 0$ corresponds to $T = 0$. In the (T, R) plane, it projects on the lower red line drawn in the figure. Its induced metric is $dR^2 + \sin^2 R d\Omega^2$. Its conformal completion is topologically a 3-sphere. If one removes its South Pole i^0, one gets $\mathbb{R}^3$.

Penrose diagram of Minkowski space

We close this subsection by noting that $\mathscr{I}^+$ and $\mathscr{I}^-$ are both cylinders. The metric induced on $\mathscr{I}^+$ or $\mathscr{I}^-$ by $d\bar{\Sigma}^2$ is degenerate and given by

$$\sin^2 R\, d\Omega^2, \qquad 0 < R < \pi. \tag{1.10}$$

The null directions are the light rays emanating from (or reaching) i^0. They have fixed angles and are parametrized by R. One can view $\mathscr{I}^+$ as the future light cone of i^0 and $\mathscr{I}^-$ as its past light cone.

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Can these conditions be justified from “reasonable” initial data ?

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which is badly represented in the Penrose diagram.

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Again, does this follow from the initial data ?

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and has also initial conditions of compact support.

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This is the situation to which we now turn.

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Here, $\overline{\gamma}_{AB} dx^A dx^B$ is the metric on the round 2-sphere (in standard (θ, φ)-variables, it reads $d\theta^2 + \sin^2\theta \, d\varphi^2$). The coefficients in the expansion are allowed to be function of time and of the angles, e.g., $\overline{\phi} = \overline{\phi}(t, x^A)$.

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To answer this question, one needs to integrate the equations of motion with given initial data on $t = 0$ (say).

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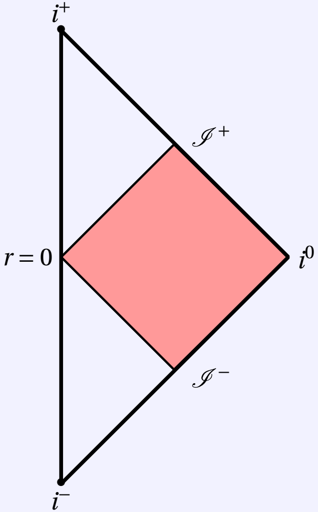

Hyperbolic coordinates cover the region outside to the light cone of the origin, i.e., the red region (without the boundary).

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If one then takes the limit $s \to 1$, one reaches the past of future null infinity "from below". Similarly, if one takes the limit $s \to -1$, one reaches the future of past null infinity "from above".

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So, for the free scalar field in hyperbolic coordinates, each order in the expansion in η^{-1} fulfills autonomous equations of motion.

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To illustrate the behaviour of the solutions, we shall consider only the cases $k=0$ and $k=1$. The general discussion can be found in

Klein-Gordon equation in hyperbolic coordinates

In order to solve these equations, we develop each of the unknown functions in spherical harmonics,

$$\phi^{(k)} = (1-s^2)^{\frac{1-k}{2}} \sum_{lm} \Theta^{(k)}_{lm}(s)\, Y_{lm}(x^A).$$

The equation satisfied by the coefficients $\Theta^{(k)}_{lm}$ is

$$(1-s^2)\partial_s^2\Theta^{(k)}_{lm} + 2(k-1)s\partial_s\Theta^{(k)}_{lm} + \Big[l(l+1)-k(k-1)\Big]\Theta^{(k)}_{lm} = 0.$$

To illustrate the behaviour of the solutions, we shall consider only the cases $k=0$ and $k=1$. The general discussion can be found in

M. Henneaux and C. Troessaert, “Asymptotic structure of a massless scalar field and its dual two-form field at spatial infinity,” JHEP **05** (2019), 147 [arXiv :1812.07445 [hep-th]]

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in terms of Legendre polynomials $P_l^{(\frac{1}{2})}(s)$ and Legendre functions of the second kind $Q_l^{(\frac{1}{2})}(s)$.

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Indeed, as one tends to null infinity along a radial null geodesics, e.g., for $\mathscr{I}^+$,

$$t = r + b \qquad (b < 0)$$

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The limiting values of (s, η) are always $(1, \infty)$ so that the information about the null geodesic (i.e., b) and where it reaches $\mathscr{I}^+$ is lost. This suggests to go to new coordinates that do not suffer from this limitation.

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In these coordinates, the Minkowskian line element becomes

$$d\Sigma^2 = \frac{\rho^2}{\left(1 - s^2\right)^2}\left(\frac{(1 - s^2)}{\rho^2} d\rho^2 + 2s\rho^{-1} ds\, d\rho - ds^2 + d\Omega^2\right)$$

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Spatial infinity is blown up as in hyperbolic coordinates and characterized by $\rho = \infty$, $s \in (-1, 1)$, with different boosted hyperplanes cutting it at different values of s.

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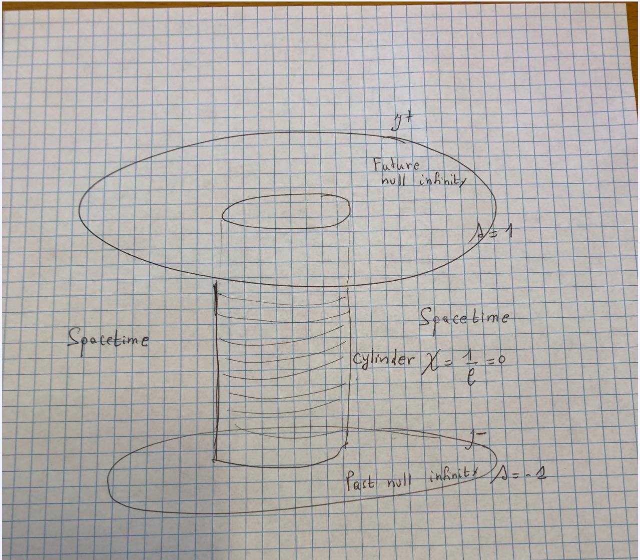
[In the original work of Friedrich, the radial coordinate being used is $\chi \equiv \rho^{-1}$ (denoted ρ in that work) instead of the coordinate ρ introduced here, so that spatial infinity is the cylinder $\chi = 0$ (with same non-degenerate induced metric in the same rescaled metric, which reads $d\tilde{\Sigma}^2 = \frac{(1-s^2)}{\chi^2} d\chi^2 - 2s\chi^{-1} ds\, d\chi - ds^2 + d\Omega^2$). Spacetime is the outside of this cylinder. With the coordinate ρ considered here, spacetime is the inside of the cylinder at spatial infinity.]

Friedrich coordinates



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The coordinate s still behaves as $s \rightarrow +1$ (respectively, -1) as one goes to $\mathscr{I}^+$ (respectively $\mathscr{I}^-$), but the new coordinate ρ assumes now values that encodes the information on "where" one reaches $\mathscr{I}^+$ (respectively $\mathscr{I}^-$).

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Thus, future null infinity $\mathscr{I}^+$ is given by $s = 1$, $\rho \in (0, \infty)$. Similarly, past null infinity $\mathscr{I}^-$ is given by $s = -1$, $\rho \in (0, \infty)$. The limitation mentioned in the previous subsection has therefore been eliminated.

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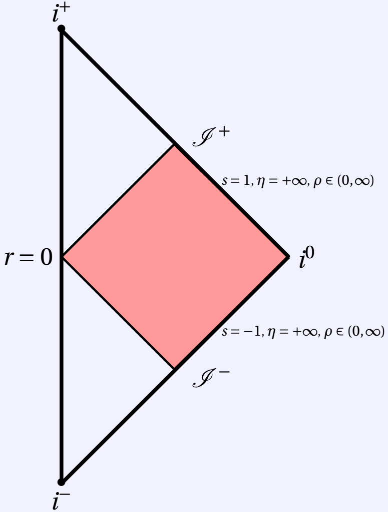
This is illustrated in the next figure.

Friedrich coordinates



Friedrich coordinates cover the same region as the hyperbolic coordinates, i.e., the region outside to the light cone of the origin (in red). Null infinity is better described in Friedrich coordinates since different points at null infinity have different coordinates - specifically, $\rho \in (0, \infty)$ is different.

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The boundaries $\rho = \infty$ of $\mathscr{I}^+$ $(s = 1)$ and $\mathscr{I}^-$ $(s = 1)$ are denoted $\mathscr{I}^+_-$ and $\mathscr{I}^-_+$ and called the critical surfaces (spheres). They are at the same time the upper and lower boundaries of the cylinder representating spatial infinity. Matching conditions connect limiting values of fields at the critical surfaces.

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Note that $s=1$, $\rho\to\infty$ corresponds to going first to $\mathscr{I}^+$ $(s=1$, $\rho>0)$ and then taking the limit to the past of $\mathscr{I}^+$ (i.e., going to spatial infinity "from above" along null infinity). Similarly, $s=-1$, $\rho\to\infty$ corresponds to going first to $\mathscr{I}^-$ $(s=-1$, $\rho>0)$ and then taking the limit to the future of $\mathscr{I}^-$ (i.e., going to spatial infinity "from below" along null infinity). Note also the fact that increasing ρ corresponds to going to spatial infinity in both cases - hence to the past for $\mathscr{I}^+$ and to the future for $\mathscr{I}^-$.

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Conversely, the limit $\rho\to\infty$, $s\in(-1,1)$ followed by $s\to 1$ (respectively -1) corresponds to going to future null infinity (respectively past null infinity) from spatial infinity.

Going to null infinity

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The link with standard retarded null coordinates (u, r) is given by

$$s = 1 + \frac{u}{r}, \qquad \rho = -2u - \frac{u^2}{r}, \qquad 1 - s^2 = -2u\frac{1}{r} + O(r^{-2})$$

where we take $u < 0$ which is relevant to the limit of going to the past of future null infinity.

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$$+ \frac{1}{r} \sum_{l,m} \Theta^{P(0)}_{lm} P_l^{(\frac{1}{2})}(1)\, Y_{lm}(x^A) + O(r^{-2}),$$

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one gets

$$Q_0^{(\frac{1}{2})}(1 + u/r) = \frac{1}{2}\left(\log(r) + \log 2 - \log(-u)\right) + o(1).$$

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Expressing ϕ in terms of u, r, we get

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$$+ \frac{1}{r} \sum_{l,m} \Theta^{P(0)}_{lm} P_l^{(\frac{1}{2})}(1)\, Y_{lm}(x^A) + O(r^{-2}),$$

Recalling $Q_0^{(\frac{1}{2})}(s) = \frac{1}{2}\log\frac{1+s}{1-s}$,
one gets

$$Q_0^{(\frac{1}{2})}(1+u/r) = \frac{1}{2}\Big(\log(r) + \log 2 - \log(-u)\Big) + o(1).$$

If the Q-branch is non-zero, the scalar field will have a term of the form $\frac{\log r}{r}$. It is interesting to see that this logarithmic branch in r is paired with a logarithmic divergence in u for the coefficient of the $\frac{1}{r}$ term.

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We have also the antipodal matching conditions $\lim_{u\to-\infty}\overline{\phi}(u,x^A) = \lim_{v\to\infty}\overline{\phi}(v,-x^A)$ relating the leading orders on the past boundary of future null infinity and on the future boundary of past null infinity at the antipodal points.

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where $P^{(\frac{3}{2})}_{l-1}(s)$ are Gegenbauer (or ultraspherical) polynomials and $Q^{(\frac{3}{2})}_{l-1}(s)$ the associated functions of the second kind.

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One has also

$$P^{(\frac{3}{2})}_n(-s) = (-)^n P^{(\frac{3}{2})}_n(s), \qquad Q^{(\frac{3}{2})}_n(-s) = (-)^{n+1} Q^{(\frac{3}{2})}_n(s).$$

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There will be terms in $\frac{1}{r}$, $\frac{\log r}{r^2}$, $\frac{1}{r^2}$ etc

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One takes $\overline{\phi}$ even and $\overline{\pi}$ odd under the antipodal map, in agreement with the conditions found from a ‘smooth” behaviour at null infinity.

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THANK YOU !