AdS|CFT & Wilson loops I

Motivations $\nearrow$ Make precision tests of AdS|CFT

How?

Because WL provide exact results which we can extrapolate to strong coupling

What is a Wilson loop? It is a closed Wilson line

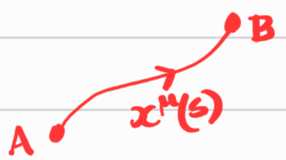
What is a Wilson line? Solution to parallel transport along a curve $\mathcal{C}$.

Given a covariant derivative $\boxed{D_\mu = \partial_\mu - i A_\mu}$

with $A_\mu = A_\mu^a T^a \in \mathrm{Lie}(G)$ gauge group

$$\begin{cases} D_\mu W = 0 \\ \oplus \\ \mathcal{C} \leftrightarrow x^\mu(s) \end{cases} \rightarrow \frac{DW}{DS} = 0 \rightsquigarrow \dot{x}^\mu(\partial_\mu W - i A_\mu W) = 0$$

$$\boxed{\frac{d}{ds} W = i A_\mu W}$$



The solution to this equation is a Wilson line

$$W[\mathcal{C}_{A\to B}] = P\, e^{i \int_A^B \dot{x}^\mu A_\mu \, ds}$$

path order

Consider an iterative solution

$$\left.\begin{aligned} &\dot{x}^\mu(\partial_\mu - i e A_\mu) W = 0 \\ &W = 1 + e W^{(1)} + e^2 W^{(2)} + \dots \end{aligned}\right\} \quad e \dot{W}^{(1)} + e^2 \dot{W}^{(2)} + \dots = i \dot{x}^\mu A_\mu (e + e^2 W^{(1)} + \dots)$$

The solution is

e^1: $\dot{W}^{(1)} = i\,\dot{x}^\mu A_\mu \longrightarrow W^{(1)}(s) = i\int_0^s ds'\, \dot{x}^\mu(s') A_\mu(s')$

e^2: $\dot{W}^{(2)} = i\,\dot{x}^\mu A_\mu W^{(1)} \rightarrow W^{(2)}(s) = i^2 \int_0^s ds'\, \dot{x}^\mu(s') A_\mu(s') \int_0^{s'} ds''\, \dot{x}^\nu(s'') A_\nu(s'')$

$\vdots$

Setting $e=1$ we arrive at

$$W[\gamma_{A\to B}] = 1 + i\int_0^s ds'\, \dot{x}^\mu(s') A_\mu(s') + i^2 \int_0^s ds' \int_0^{s'} ds''\, \underbrace{\left(\dot{x}^\mu(s') A_\mu(s')\right)\left(\dot{x}^\nu(s'') A_\nu(s'')\right)}_{s' > s''}$$

$$+\, i^3 \int_0^s ds' \int_0^{s'} ds'' \int_0^{s''} ds'''\, \underbrace{A(s') A(s'') A(s''')}_{\text{path ordered}} + \dots$$

$A(s) = \dot{x}^\mu(s) A^\mu(s)$

This expression is formally written as

$$P e^{i\int_0^s ds' A(s')} = 1 + i\int_0^s ds'\, A(s') + \frac{i^2}{2!} \int_0^s ds' \int_0^s ds''\, P\underbrace{[A(s') A(s'')]}_{2!\ \text{terms}}$$

$$\dots + \frac{i^n}{n!} \int_0^s ds \dots \int_0^s ds^{(n)}\, P\underbrace{[A(s) \dots A(s^{(n)})]}_{n!\ \text{terms}}$$

Completely analogous to the evolution operator (T-order) in QFT

Note: it is immediate to see that matrices are rearranged in reverse order under hermitian conjugation hence

$$\boxed{\left(W[\gamma_{A\to B}]\right)^\dagger = W[\gamma^{-1}_{B\to A}] \rightarrow W[\gamma_{A\to B}] \cdot W[\gamma_{A\to B}]^\dagger = 1}$$

What happens under gauge transfs?

$$A_\mu \to A'_\mu = g \cdot A_\mu \cdot g^{-1} - i\, g\, \partial_\mu g^{-1}$$

$$\Rightarrow \boxed{D'_\mu = g \cdot D_\mu \cdot g^{-1}} \qquad g = g(x) \in G$$

G → gauge group

$$\Rightarrow \boxed{W'[\gamma_{A\to B}] = g(B) \cdot W[\gamma_{A\to B}] \cdot g^{-1}(A)}$$

kind of gauge covariant.

It was Schwinger ('59) who introduced Wilson lines as natural (non-local) operators in order to perform point splitting regul. in a gauge invariant way:

$J_\mu(x) = \bar\psi(x)\, \gamma_\mu\, \psi(x)$ is invariant under $\psi(x) \to \psi'(x) = g(x)\, \psi(x)$

$$\bar\psi'(x) = \bar\psi(x) \cdot \underbrace{g^\dagger(x)}_{= g^{-1}(x)}$$

However when making a point splitting

$\cancel{J_\mu^{(\varepsilon)}(x) \equiv \bar\psi(x+\tfrac{\varepsilon}{2})\, \gamma_\mu\, \psi(x-\tfrac{\varepsilon}{2})}$ is not gauge invariant

$$\to \boxed{\bar\psi(x+\tfrac{\varepsilon}{2})\, \underbrace{g^{-1}(x+\tfrac{\varepsilon}{2})\, g(x-\tfrac{\varepsilon}{2})}_{\neq 1}\, \gamma_\mu\, \psi(x-\tfrac{\varepsilon}{2})}$$

gauge factors do not cancel

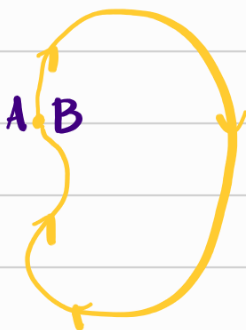
This is remedied introducing a Wilson line

$$\boxed{J_\mu^{(\epsilon)} \equiv \bar{\Psi}\left(x+\frac{\epsilon}{2}\right) \underbrace{e^{i\int_{x-\frac{\epsilon}{2}}^{x+\frac{\epsilon}{2}} ds\, A(s)}}_{\text{Compensates the gauge factors}} \gamma_\mu \Psi\left(x-\frac{\epsilon}{2}\right)}$$

& properly accounts for anomalies!

Why a loop?

Because if $A = B \rightsquigarrow$ closed loop



$$W'[\mathcal{C}] = g(A)\cdot W[\mathcal{C}]\cdot g^{-1}(A)$$; gauge covariant!

Then taking a trace

$$\boxed{W[\mathcal{C}] = \mathrm{tr}\, P e^{i\oint_{\mathcal{C}} A}}$$ is gauge invariant

The trace of the holonomy of the gauge connection is a physical observable.

Inside the path integral

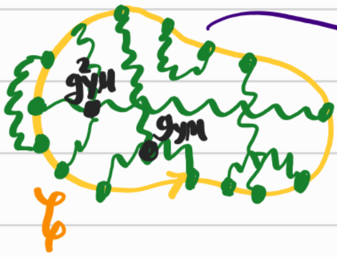
$$\langle W[\mathcal{C}]\rangle = \int \mathcal{D}A_\mu\, e^{iS_{YM}} \underbrace{\mathrm{tr}\, e^{i\oint A}}_{\text{Viewed as coupling } A_\mu \text{ to an external particle}} \leftrightarrow e^{i\int J^\mu A_\mu}$$

supported on line; i.e. with

$$J^\mu(x) = q\int ds\, \delta^{(4)}(x-y(s))\frac{dy^\mu}{ds}$$

(point charge current)

At quantum level, the WL V.E.V involves



we expect divergences when points coincide

More precisely

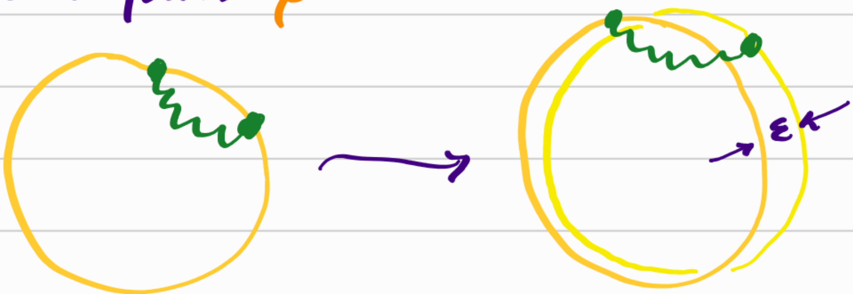
$$\langle W[\gamma]\rangle = N\int \mathcal{D}A_\mu \; e^{-\frac{1}{g_{YM}^2}\int F_{\mu\nu}^2} \, \mathrm{tr}\, P \, e^{i\oint_\gamma A_\mu \dot{x}^\mu ds}$$

$$\sim 1 + i\oint_\gamma \langle \cancel{A_\mu} \rangle \dot{x}^\mu ds$$

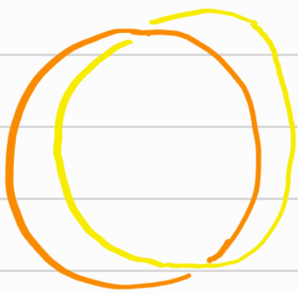
$$+ \frac{i^2}{2!}\oint_\gamma ds \oint_\gamma ds' \left\langle P\left\{ \overbrace{A_\mu(x(s))\, A_\nu(x(s'))}^{\text{Wick contract}} \right\}\right\rangle$$

$$+ \dots$$

So a regularization is needed, e.g computing the integrals displacing the contours γ



But topological issues arise, a.k.a. framing $\longleftrightarrow$



linked contours

cf Witten's Jones polynomial is 3d Chern+Simons

Note: • in QED $U(1)$ abelian gauge theory, irreps are classified by charge $q \in \mathbb{Z} \rightsquigarrow q = ne$ ← fundamental charge

Perturbation theory is performed in powers of e

• In non-abelian gauge theories quark charge ⟷ $SU(N)$ irrep

g_{YM} is fixed and used for making perturbative expansion.

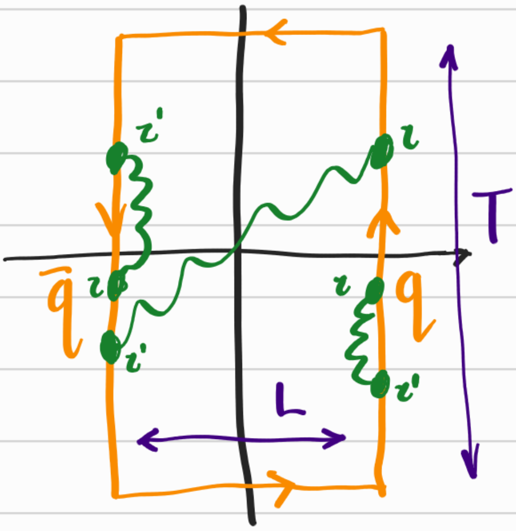
$$W[\mathcal{C}] = \mathrm{tr}\, P\, e^{i \oint_{\mathcal{C}} A_\mu dx^\mu}$$

$$A_\mu = A_\mu^a T^a$$

what we modify is ← choice of irrep ⟷ particle charge

T^a → fund, antifund, k-symm, k-antisym, Young Tableaux

Wilson Contour & Confinement criterium



Consider computing this loop in QED (abelian/Gaussian) Integration is immediate:

$$\langle W[\boxed{\uparrow}^{T}_{L}]\rangle = N\int \mathcal{D}A_\mu \; e^{-\int F_{\mu\nu}^2 d^4x} \; e^{ie\oint A_\mu \dot{x}^\mu ds}$$

$$= N \; e^{-\int dx\,dx' \frac{J^\mu(x)\, J_\mu(x')}{|x-x'|^2}}$$

$$= N \; e^{-e^2\oint_\gamma ds \oint_\gamma ds' \frac{\dot{x}(s)\cdot \dot{x}(s')}{|x(s)-x(s')|^2}}$$

From classical electromagnetism we know the result will involve:

- self energy  → divergent for coincident points
  → proportional to loop's length extension
  → can be cancelled by counterterm $e^{-\Delta\oint_\gamma ds}$
- Interaction energy 

→ $\boxed{V_{q\bar{q}}(L) = -\frac{e^2}{L}}$ → Coulomb potential

We conclude that for QED, WL for rectangular γ substracting obvious divergencies

$$\langle W[\square] \rangle \sim e^{-V_{q\bar{q}}(L)\cdot T} = e^{e^2 T/L}$$

Note: $V_{q\bar{q}} = \dfrac{f(g^2_{YM}, N)}{L}$ for conformal theories with AdS duals.

dictated by scale invariance of Maxwell's theory

With this insight in mind, Kenneth introduced

$$\underbrace{e^{i\oint A_\mu dx^\mu}}_{QED} \rightsquigarrow \boxed{W[\gamma] \equiv \mathrm{tr}\, P e^{i\oint_\gamma A_\mu dx^\mu}}$$

and proposed Wilson criterium, i.e.

$$\boxed{\text{Linear Confinement} \equiv \text{Wilson Loop Area Law}}$$

Meaning that

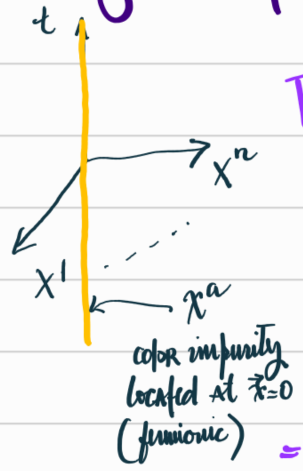
$$\langle W[\square] \rangle \sim e^{-\text{Area}} \sim e^{-\alpha L T}$$

dimensionfull "QCD scale" QCD string tension

$$\rightarrow \boxed{V_{q\bar{q}} = \alpha L}$$ linear Confinement

Wilson loop as a defect

The Wilson loop operator can be understood as a (dynamical) localized impurity inserted into the ambient theory



$$W[\gamma_{A\to B}] = 1 + i\int_0^s ds'\, \dot{x}^\mu(s')A_\mu(s') + i^2\int_0^s ds'\int_0^{s'} ds''\, \underbrace{\left(\dot{x}^\mu(s')A_\mu(s')\right)\left(\dot{x}^\nu(s'')A_\nu(s'')\right)}_{s'>s''}$$

$$+\, i^3\int_0^s ds'\int_0^{s'} ds''\int_0^{s''} ds'''\, \underbrace{A(s')A(s'')A(s''')}_{\text{path ordered}} + \ldots$$

$$= 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!}\int_0^s ds' \ldots \int_0^s ds^{(n)}\, \theta(s'-s'')\,\theta(s''-s''')\ldots\theta(s^{(n-1)}-s^{(n)})\, A(s')\ldots A(s^{(n)})$$

The idea is to interpret $\theta(s-s')$ as the propagator of an auxiliary "fundamental" fermion field $\chi(s)$ living on the WL contour interacting with ambient gauge theory as

$$\langle W[\gamma_{A\to B}]\rangle = N\int \mathcal{D}A_\mu\, \mathcal{D}\chi^a \mathcal{D}\bar{\chi}_a\, \chi(B)\,\bar{\chi}(A)\, e^{-S_{YM} - \underbrace{\int_A^B ds\, \bar{\chi}(\partial_s - iA(s))\chi}_{\text{Gaussian } \chi^a \text{ theory}}}$$

$$\boxed{W[\gamma_{A\to B}] = \langle \chi(B)\,\bar{\chi}(A)\rangle_A}$$

Then the Wilson line $\gamma_{A\to B}$ is the 2-pt function of the 1-dim defect theory defined on it. Assuming now the loop to be closed: $A=B$

$A \equiv x^\mu(0) = x^\mu(2\pi) \equiv B$ & $\chi^a(0) = -\chi^a(2\pi)$ anti-periodic fermions

$$\boxed{\langle W[\gamma]\rangle = N\int \mathcal{D}A_\mu\, \mathcal{D}\chi^a \mathcal{D}\bar{\chi}_a\, \chi^a(2\pi)\,\bar{\chi}_a(0)\, e^{-S_{YM} - S_{defect}}}$$

Ladder diagrams resummation WL

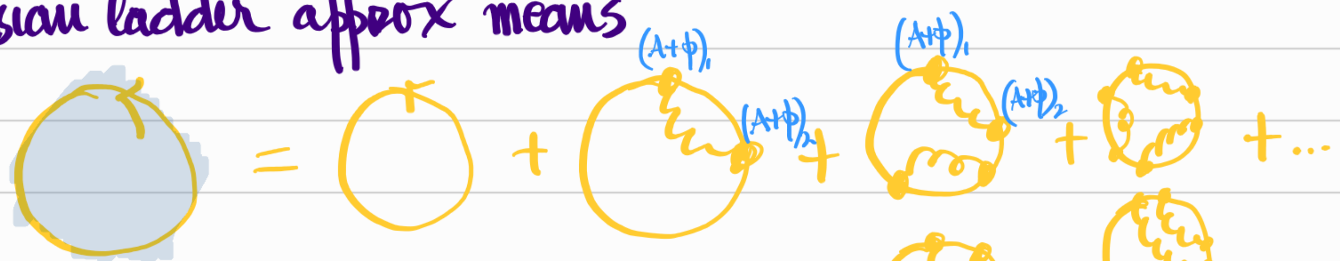
Planar diagrams ladder resummation display non trivial coupling dependence & moreover capture the full the result for the circular susy WL.

$$W[\gamma] = \frac{1}{N}\, \mathrm{tr}\, P \exp\left[\oint_\gamma \left(i A_\mu \dot{x}^\mu + \vec{\phi}\cdot\vec{n}\, |\dot{x}| \right) ds \right]$$

γ: $x^\mu(s) = R(\cos s, \sin s, 0, 0)$

$\vec{n} = (1,0,0,0,0,0)$ $\qquad s \in (0, 2\pi)$

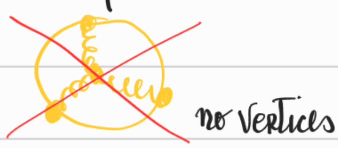
The gaussian ladder approx means



taking into account

$$\langle A_\mu^a(x) A_\nu^b(y) \rangle = \frac{g_{YM}^2}{4\pi^2} \frac{\delta_{\mu\nu}\, \delta^{ab}}{|x-y|^2}$$

$$\langle \phi_i^a(x) \phi_j^b(y) \rangle = \frac{g_{YM}^2}{4\pi^2} \frac{\delta_{ij}\, \delta^{ab}}{|x-y|^2}$$



$$\left\langle \left(i A_\mu \dot{x}^\mu + \vec{\phi}\cdot\vec{n}\, |\dot{x}| \right)_{1\,j}^{\;i} \left(i A_\mu \dot{x}^\mu + \vec{\phi}\cdot\vec{n}\, |\dot{x}| \right)_{2\,l}^{\;k} \right\rangle$$

$$= \frac{1}{N}\, \delta^i_l\, \delta^k_j\, \frac{\lambda}{8\pi^2} \underbrace{\frac{|\dot{x}|_1 |\dot{x}|_2 - \dot{x}_1 \cdot \dot{x}_2}{|x_1 - x_2|^2}}_{\text{constant for the circle } = 1/2}$$

$i, j, k, l = SU(N)$ indices

Contractions give a constant. We need to count the number of diagrams

$\rightsquigarrow$ $N_{n+1} = \sum_{k=0}^{n} N_k N_{n-k}$ $\oplus$ $N_0 = 1$ $\rightsquigarrow$ $\boxed{\langle W[\bigcirc] \rangle = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda})}$

Wilson loops in susy gauge theories

We will be interested in WL preserving some susi's

$\Rightarrow$ will involve additional couplings to scalars & fermions

$$W[\gamma] = \mathrm{tr}\, P \exp\left(i \oint_\gamma (A+\phi)\right)$$

Susy gauge theories

The two paradigmatic examples of AdS/CFT are

- $\mathcal{N}=4$ SYM $SU(N_c)$, $d=3+1$ (32 susies) $\longleftrightarrow$ IIB string theory on $AdS_5 \times S^5$ $\oplus$ $F_{(5)} \neq 0$

- $\mathcal{N}=6$ Chern Simons Matter $U_k(N_c) \times U_{-k}(N_c)$, $d=2+1$ (24 susies) $\longleftrightarrow$ IIA strings in $AdS_4 \times \mathbb{CP}^3$

Assuming this to be true we need to be more precise, i.e. we need to give a prescription about how the two theories relate.

$$S = \int (\partial\phi)^2 + g\, V(\phi)$$

$$S = \int (\partial\psi)^2 + \tilde{g}\, \tilde{V}(\psi)$$

$$\boxed{g = \frac{1}{\tilde{g}}}$$

Strong/Weak

AdS dictionary:

1. Parameters:

$\mathcal{N}=4$: $(g_{YM}, N_c) \to (\lambda = g_{YM}^2 N_c, \frac{1}{N_c})$ a.k.a 't Hooft coupling

IIB strings on $AdS_5 \times S^5$ $(\alpha', \ell_{AdS}^2, g_s) \to T_{eff} = \frac{\ell_{AdS}^2}{\alpha'}$

$$\boxed{\lambda = \left(\frac{\ell_{AdS}^2}{\alpha'}\right)^2} \qquad \boxed{\frac{\lambda}{4\pi N_c} = g_s}$$

Maldacena '97

$g_{YM}^2 = g_s$

The two theories are:

$$S_{SYM} = \int d^4x \, \mathrm{Tr}\Big\{ -\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2}(D_\mu \phi^i)^2 + \bar{\Psi}^a \not{D} \Psi_a - g_{YM}^2 [\phi^i, \phi^j]^2 - g_{YM} \bar{\Psi}^a (\rho_i)_a{}^b [\phi^i, \Psi_b] \Big\}$$

susy index $a=1,..4$

$SO(6)$ gamma matrices $\{\rho_i, \rho_j\} = 2\delta_{ij}$, $i,j=1,..6$

with $(F_{\mu\nu}, \Psi^a, \phi^i) \in \mathrm{Lie}(SU(N_c))$

and

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + g_{YM}^2 [A_\mu, A_\nu]$$

$$D_\mu = \partial_\mu + g_{YM} [A_\mu, \cdot\,]$$

&

$$S_{IIB} = \frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{h}\, h^{\alpha\beta} G_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu + \text{fermions}$$

$G_{\mu\nu} = \ell_{AdS}^2 \, (AdS_5 \times S^5)$

Note: $[\alpha'] = L^2$

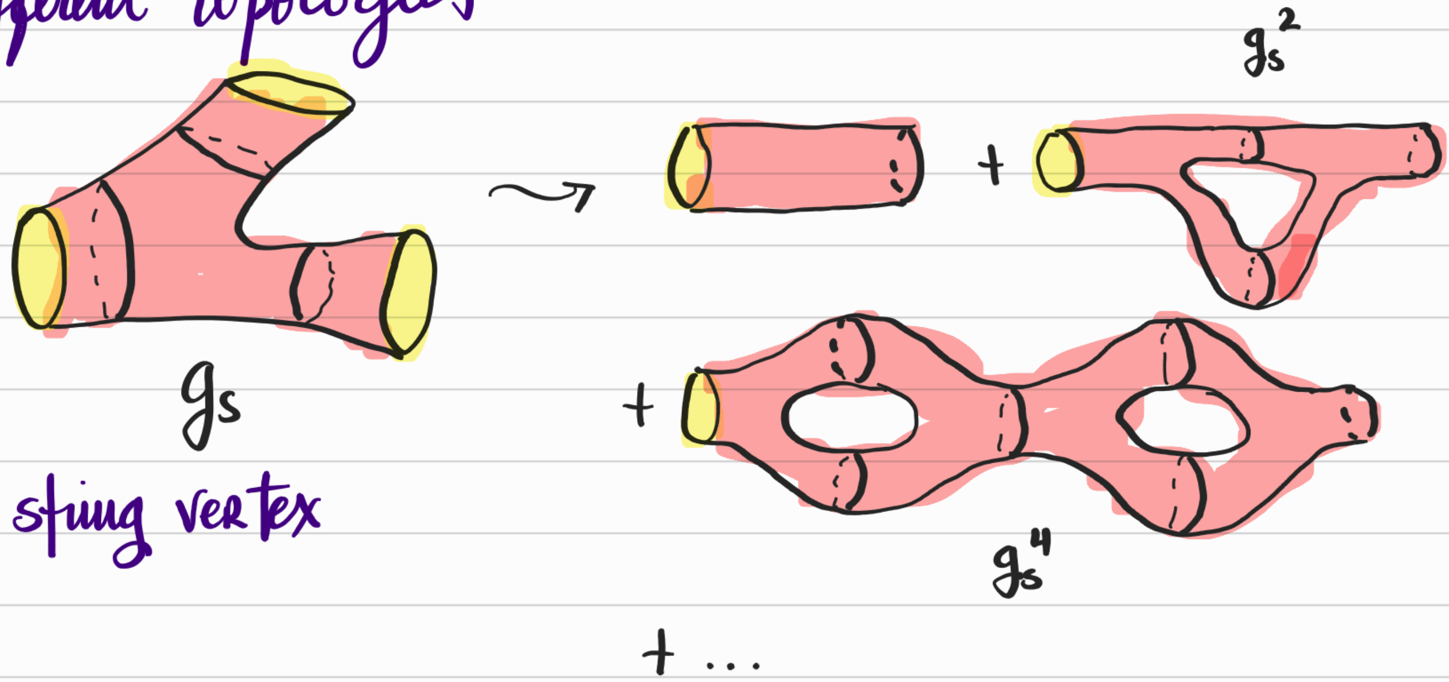
$\rightsquigarrow$ $S_{polyak} = \frac{\ell_{AdS}^2}{\alpha'} \int d^2\sigma \ldots$

$\sim \frac{1}{\hbar} \to$ Semiclassical limit on string side

strong coupling in gauge theory

$$\boxed{\frac{\ell_{AdS}^2}{\alpha'} \gg 1}$$

the additional coupling in string theory weights the different topologies



The AdS/CFT correspondence proposes

$$\boxed{\begin{aligned} \text{Hilbert}_{CFT} &= \text{Hilbert}_{AdS} \\ \int \mathcal{D}A_\mu \mathcal{D}\phi^i \mathcal{D}\psi^a e^{-S_{SYM}} &= \int \mathcal{D}h_{\alpha\beta} \mathcal{D}X^\mu e^{-S_{pol}} \\ Z_{CFT}[\lambda, N_c] &= Z_{AdS}\left[\frac{\ell_{AdS}^2}{\alpha'}, g_s\right] \end{aligned}}$$

What is a CFT & Why study them?

AdS/CFT & WL II

$$\boxed{\text{CFT} = \text{Teoría cuántica de campos invariante bajo el grupo conforme}}$$

Grupo conforme

Es el conjunto de transformaciones (continuas) $x \to x' = x'(x)$ que preservan ángulos, i.e. dejan invariante la métrica a menos de un factor de escala

$$ds^2 = (dx^\mu)^2 \longrightarrow ds'^2 = \underbrace{(dx'^\mu)^2}_{\substack{\text{generalization} \\ \text{of } x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu}} = \underbrace{\Omega^2(x)}_{\text{local scale transformation}} ds^2$$

$\mu = 1, \dots d$

$x'^\mu = \underbrace{\underbrace{\Lambda^\mu{}_\nu x^\nu}_{\text{Lorentz } (M_{\mu\nu})} + \underbrace{a^\mu}_{\text{translation } (P_\mu)}}_{\text{Poincare}}$ } has $\Omega(x) = 1$

The condition can be equivalently written as

$$\boxed{g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x) = \Omega^2(x)\, g_{\mu\nu}(x)}$$

Plugging

$g_{\mu\nu} \to \eta_{\mu\nu}$ Lorentzian $SO(2,d)$

$g_{\mu\nu} \to \delta_{\mu\nu}$ Euclidean $SO(1,d+1)$

To find the possible diffeos we write

$$x'^{\mu} = x^{\mu} + \epsilon\, \xi^{\mu}(x)$$

$$\Omega(x) = 1 + \epsilon\, \sigma(x)$$

$\xi_\mu \equiv \eta_{\mu\nu}\xi^\nu$

$$\Rightarrow (*) \quad \boxed{\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = 2\sigma(x)\, \eta_{\mu\nu}}$$

The fancy name for this eqn is conformal Killing eqn

$$\boxed{\mathcal{L}_\xi\, g_{\mu\nu} = 2\sigma\, g_{\mu\nu}}$$

To solve (*) we take the trace $\rightsquigarrow$ $\sigma = \frac{1}{d}\, \partial\cdot\xi$

Manipulating the eqn $\rightsquigarrow$

$$\left(\eta_{\mu\nu}\Box + (d-2)\,\partial_\mu\partial_\nu\right)\partial\cdot\xi = 0 \quad (1)$$

and an additional trace $\rightsquigarrow$ $(d-1)\,\Box\,(\partial\cdot\xi) = 0$ (2)

$\Rightarrow$ if $d \neq 2$ (1) + (2) $\Rightarrow$ $\boxed{\partial_\mu\partial_\nu(\partial\cdot\xi) = 0}$ $d\neq 2$

(*) $\Rightarrow$ $\partial\cdot\xi$ = linear in x^μ $\Rightarrow$ ξ^μ = at most quadratic in x

$$\boxed{\xi^\mu(x) = \underbrace{a^\mu}_{\text{translation}} + \underbrace{\omega^\mu{}_\nu x^\nu}_{\text{Rotation/Boost}} + \underbrace{\kappa\, x^\mu}_{\text{dilatation}} + \underbrace{b^\mu x^2 - 2x^\mu (b_\nu x^\nu)}_{\text{special conformal}}}$$

$\omega_{\mu\nu} = -\omega_{\nu\mu}$, $\omega_{\mu\nu} \equiv \eta_{\mu\rho}\,\omega^\rho{}_\nu$

Total # of generators/parameters is

$$\frac{1}{2}(d+2)(d+1)$$

$a^\mu, \omega^\mu{}_\nu \rightsquigarrow$ Killing vectors ($\Omega = 1$) $\Big\}$ Conformal Killing Vector

$\kappa, b^\mu \rightsquigarrow \Omega \neq 1$

<u>Note:</u> $d=2$ is special $\rightsquigarrow$ Conformal group becomes ∞-dim

We can finite the finite form, a.k.a integral curves of the vector field (exp map) by solving

$$\frac{dx^\mu}{dt} = \xi^\mu(x)$$

$\oplus$

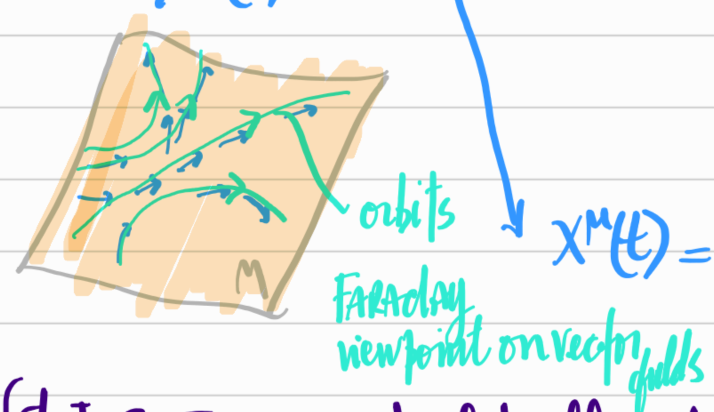
$$x^\mu(0) = x^\mu$$

$\to x^\mu(t) = x^\mu + t a^\mu$

$\to x^\mu(t) = \Lambda^\mu{}_\nu(t)\, x^\nu, \quad \Lambda^\mu{}_\nu(t) = \left(e^{t\omega}\right)^\mu{}_\nu$

$\to x^\mu(t) = \underbrace{e^{\kappa t}}_{\lambda = \text{scale factor}} x^\mu \to \Omega(x) = \lambda$

$\to x^\mu(t) = \dfrac{x^\mu + t b^\mu x^2}{1 + 2t\, b\cdot x + t^2 b^2 x^2} \to \Omega(x) = \dfrac{1}{(1 + 2 b\cdot x + b^2 x^2)}$



<u>Note I:</u> SCT are not globally defined on $\mathbb{R}^d / \mathbb{R}^{d-1,1}$ since

$$x'^\mu = \frac{x^\mu + b^\mu x^2}{1 + 2\, b\cdot x + b^2 x^2} \quad \text{maps} \quad x^\mu = -\frac{b^\mu}{b^2} \to x' = \infty \;!!$$

The appropriate definition involves working with the conformal compactification (adding points @ ∞ $\rightsquigarrow$ well defined on sphere)

Note II: it is easy to show that

$$\underbrace{I : x^\mu \to x'^\mu = \frac{x^\mu}{x^2}}$$

. Maps origin $\to \infty$!
. Properly defined in S^d (sphere)
$\mathbb{R} \times S^{d-1}$ (ESU)

SCT = Inversion $\circ$ Translation $\circ$ Inversion

So you may find the conformal group defined as

Conformal Group = Poincare $\oplus$ scale transformations $\oplus$ Inversion

We can write the differential ops:

$$P_\mu = -i\partial_\mu \quad , \quad M_{\mu\nu} = i\left(x_\mu \partial_\nu - x_\nu \partial_\mu\right)$$

$$D = -i x^\mu \partial_\mu \quad , \quad K_\mu = -i\left(x^2 \partial_\mu - 2x_\mu x^\nu \partial_\nu\right)$$

which satisfy the algebra

$$[D, P_\mu] = i P_\mu \quad , \quad [D, K_\mu] = -i K_\mu$$

$$[K_\mu, P_\nu] = -2i\left(\eta_{\mu\nu} D - M_{\mu\nu}\right)$$

$$[K_\rho, M_{\mu\nu}] = i\left(\eta_{\rho\mu} K_\nu - \eta_{\rho\nu} K_\mu\right) \quad \& \text{ idem } K \to P$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = i\left(\eta_{\nu\rho} M_{\mu\sigma} + \dots\right)$$

This is the SO(2,d) algebra by making

$$\underbrace{S_{\mu\nu} = M_{\mu\nu} \,,\; S_{-1d} = D \,,\; S_{-1\mu} = \tfrac{1}{2}(K_\mu + P_\mu) \,,\; S_{d\mu} = \tfrac{1}{2}(P_\mu - K_\mu)}_{\equiv\, S_{ab}} \quad \text{with } a,b = (-1, \underbrace{0, 1 \dots, d-1}_{\text{Minkowski}}, d) \text{ with } \eta_{ab} = \mathrm{diag}(\overbrace{-\,-}^{2}, \overbrace{+\,+ \dots +}^{d})$$

$$\boxed{[S_{ab}, S_{cd}] = i\left(S_{ac}\eta_{bd} - S_{bc}\eta_{ad} + \dots\right)}$$

SO(2,d) algebra (Lorentz)

we can picture the CKillings inside S_{ab} as

$$S_{ab} = \begin{array}{c} \\ -1 \\ 0 \\ \vdots \\ d-1 \\ d \end{array} \begin{array}{c} \begin{array}{ccccc} -1 & 0 & \cdots & d-1 & d \end{array} \\ \left(\begin{array}{c|c|c} 0 & \frac{1}{2}(P_\mu + K_\mu) & D \\ \hline \vdots & & \\ \vdots & M_{\mu\nu} & \frac{1}{2}(K_\mu - P_\mu) \\ \hline \vdots & \cdots & 0 \end{array}\right) \end{array}$$

In the euclidean case $\eta_{\mu\nu} = \delta_{\mu\nu}$ we obtain $SO(1,d+1)$

and the C Killings are oriented as:

$$\underbrace{J_{ij} = M_{ij}, \quad J_{0i} = \frac{1}{2}(P_i - K_i), \quad J_{-0} = D, \quad J_{-i} = \frac{1}{2}(P_i + K_i)}_{\equiv J_{ab}}$$

$i,j = 1,..d$

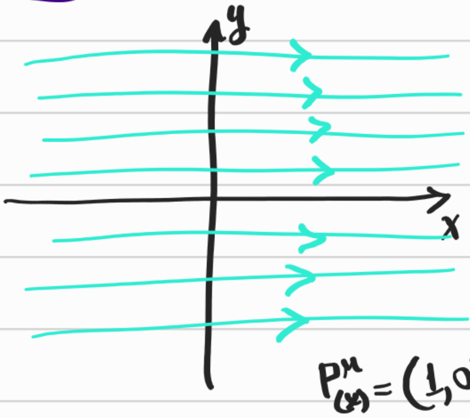
$a,b = (-,0,1,..d)$ Euclidean

$\eta_{ab} = (-,\underbrace{+,+..+}_{d+1})$

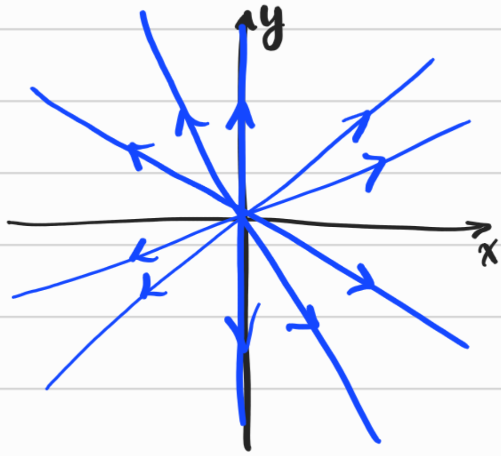
$$J_{ab} = \begin{array}{c|ccc} & -1 & 0 & i \\ \hline -1 & & & \\ 0 & & & \\ i & & & \end{array} \left(\begin{array}{c|c|c} 0 & D & \frac{1}{2}(P_i + K_i) \\ \hline -D & 0 & \frac{1}{2}(P_i - K_i) \\ \hline \vdots & \vdots & M_{ij} \end{array}\right)$$

CKV orbits / integral curves

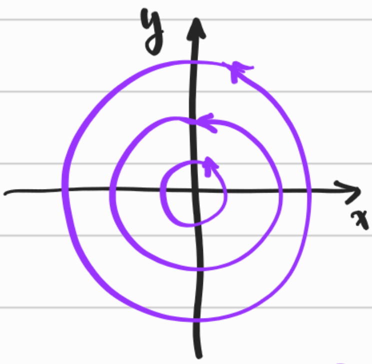
Euclidean:



$P^\mu_{(x,y)} = (1,0)$



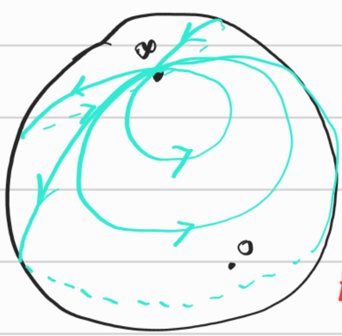
$D^\mu = (x,y)$



$M^\mu_{(x,y)} = (-y,x)$

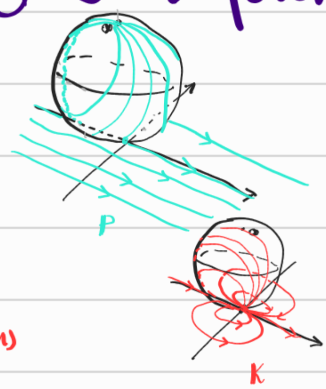
$K^\mu_{(y)}$ for $b_\mu = (0,1)$

- SCT / K^μ MAP straight lines to circles & VICEVERSA (hyperbola in Lorentz)
- If we think about the action of P_μ & K_μ on the sphere they act in an identical way wrt origin & "∞" interchanged

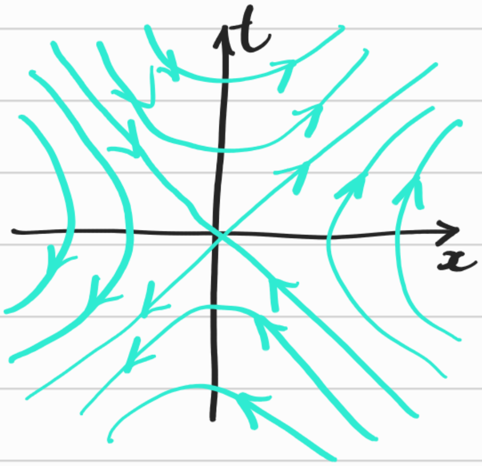


$P_{(\mu)}$

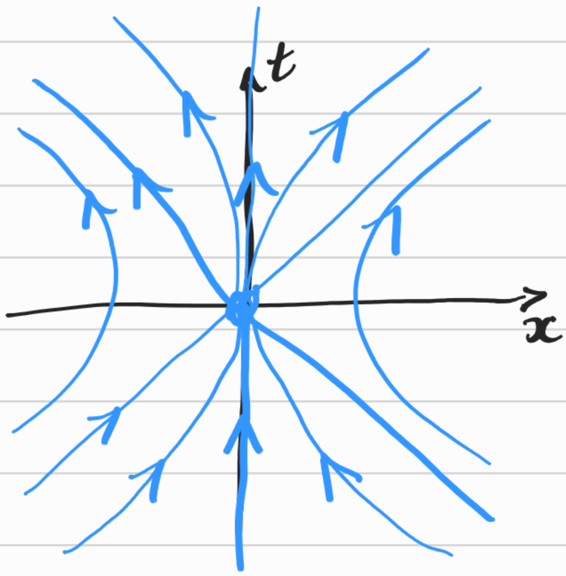
$K_{(\mu)}$



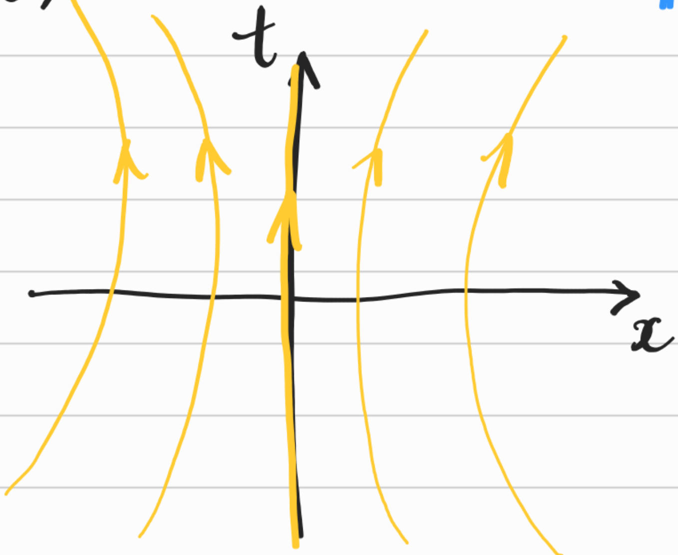
Lorentzian: P_μ, D idem



$M_{(tx)}$



$K_{(t)}$



$K_{(t)} + P_{(t)}$

Representations of the conformal group on fields

What are the good building blocks? a.k.a "Primary fields"

From the defn of the conformal group we have

Local scale factor

Local Lorentz Rotation

$$\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\eta_{\rho\sigma}(x) = \Omega^2(x)\,\eta_{\mu\nu}(x) \Rightarrow \boxed{\frac{\partial x'^\mu}{\partial x^\rho} = \Omega(x)\,R^\mu{}_\nu(x)}$$

where

$$R^\mu{}_\alpha(x)\,R^\nu{}_\beta(x)\,\eta_{\mu\nu} = \eta_{\alpha\beta}$$

$$\left|\frac{\partial x'}{\partial x}\right| = (\Omega(x))^d$$

Primary field: it is a field which under $x \to x' = x'(x)$ a Conf transf transforms as

$$x \to x' \Rightarrow \phi'(x') = \frac{1}{(\Omega(x))^{\Delta_\phi}} \phi(x)$$ SCALAR PRIMARY

$$\boxed{\phi'_I(x') = \frac{1}{(\Omega(x))^{\Delta_\phi}} \rho_I{}^J(R(x)) \phi_J(x)}$$ general primary tensor

Lorentz indices

new feature brought by scale invariance

irrep of Lorentz

Δ_ϕ = scale dimension of ϕ

Comments: • ϕ_I are our usual fields in QFT (which transform with $\Omega = 1$)

$x' = \Lambda x \rightsquigarrow$

$$\phi'(x') = \phi(x)$$

$$A'_\mu(x') = (e^{\omega})_\mu{}^\nu A_\nu(x)$$

$$\psi'(x') = e^{\omega_{\mu\nu}\Gamma^{\mu\nu}} \psi(x)$$

with $\Lambda^\mu{}_\nu = (e^\omega)^\mu{}_\nu$

since $\frac{\partial x'}{\partial x} = \Lambda = R$

If we compute (*) infinitesimally $x'^\mu = x^\mu + \varepsilon \xi^\mu(x)$ with $\delta\phi(x) = \phi'(x) - \phi(x)$

$$\delta\phi_I = -\xi^\mu \partial_\mu \phi_I - \sigma_\xi \Delta_\phi \phi_I + \frac{1}{2} \omega_\xi^{\mu\nu} (M_{\mu\nu})_I{}^J \phi_J$$

Classical scale dimensions

- Important concept to classify the possible behavior of field theory at quantum level.
- Leads to classification of interactions as relevant / marginal / irrelevant

$$S = \int d^dx \left[(\partial_\mu \phi)^2 - V(\phi) \right]$$

$$[S] = L^0 \qquad \left.\begin{array}{l}[d^dx] = L^d \\ [\partial_\mu] = \frac{1}{L}\end{array}\right\} \quad L^0 = L^d \frac{1}{L^2} [\phi]^2$$

$$\Rightarrow [\phi] = L^{-\frac{(d-2)}{2}}$$

classical scale dimension of a scalar field

$$\boxed{\Delta_\phi = \frac{d}{2} - 1}$$

d	Δ_ϕ
2	0
3	1/2
4	1
5	3/2

In a similar way,

$$S = \int d^dx \, \bar{\Psi} \not{\partial} \Psi \rightsquigarrow [\Psi] = L^{-(d-1)/2} \qquad \boxed{\Delta_\Psi = \frac{d-1}{2}}$$ fermion

$$S = \int d^dx \, (\partial_\mu A_\nu)^2 \rightsquigarrow \qquad \boxed{\Delta_{A_\mu} = \frac{d}{2} - 1}$$ vector "scalar

$$[V(\phi)] = L^{-d}$$

- $V = \lambda \phi^4 \rightsquigarrow [\lambda][\phi]^4 = L^{-d} \rightsquigarrow [\lambda] = L^{-(d-4(\frac{d}{2}-1))}$
$$= L^{d-4} = L^{-(4-d)}$$
λ is dimensionless in $d=4$

- $V = g_6 \phi^6 \rightsquigarrow [g_6] = L^{-(d-6(\frac{d}{2}-1))} = L^{2d-6} = L^{-(6-2d)}$

g_6 is dimensionless in $d=3$

Gauge coupling:

$$D_\mu = \partial_\mu + i g_{YM} A_\mu$$
$$[D_\mu] = [\partial_\mu] = L^{-1}$$
$$[g_{YM}][A] = L^{-1} \rightsquigarrow [g_{YM}] = L^{-(1-(\frac{d}{2}-1))}$$
$$= L^{-(2-\frac{d}{2})}$$

g_{YM} is dimensionless in $d=4$

Relevant & irrelevant couplings

$$[g] = E^{\alpha}$$

$\alpha > 0 \Rightarrow$ Relevant in IR

$\alpha < 0 \Rightarrow$ Irrelevant in IR

$\alpha = 0 \Rightarrow$ MARginal

$\shortparallel$

classically dimensionless

$\Delta_g \gtrless d$

irrelevant ($>$), relevant ($<$)

$$V(\phi) = \underbrace{m^2}_{} \phi^2$$

relevant coupling constant $\rightsquigarrow$ important in IR

$$E(p) = \sqrt{p^2 + m^2} \xrightarrow[\text{IR}]{p \to 0} E \sim m \quad \text{vs.} \quad p_{m^2=0}$$

"RG flow"

$$p \to \infty \; (\text{UV}): \quad E \sim p$$

Fermi coupling: $V(\psi) = G_F \, \psi^4 \rightarrow [G_F] = L^{-\left(d - 4\left(\frac{d-1}{2}\right)\right)}$

$$[G_F] = L^{-(2-d)}$$

$[G_F] = E^{-2}$ in $d=4$ $\rightsquigarrow$ $\boxed{G_F \sim \frac{1}{M_Z^2}}$ electroweak scale

An amplitude will involve at the perturbative level

$$\mathcal{A} = G_F \times + G_F^2 \times + \dots$$

dimensionless $\Rightarrow$

$$\frac{p^2}{M_Z^2} \tilde{\times} + \frac{p^4}{M_Z^4} \tilde{\times} + \dots \xrightarrow[p \to 0]{} 0$$

IR irrelevant

Marginal couplings receive quantum corrections which can make them

classically marginal coupling $[g_{YM}]$

- marginally irrelevant e_{QED}
- exactly marginal g_{YM} $\mathcal{N}=4$
- marginally relevant g_{QCD}

This statements are usually phrased in terms of the behavior of the β-function.

$$\beta(g) = \frac{d\,g}{d\ln\mu}$$

energy scale

- $\beta(e) > 0$
- $\boxed{\beta(g_{YM}) = 0}$ CFT
- $\beta(g_{QCD}) < 0$ Asymptotic freedom

Comment II: • Scale dimensions receive corrections at quantum level, a.k.a "anomalous dimensions"

$$\Delta_\phi = \Delta_\phi^{(cl)} + \gamma$$

Anomalous dimension

AdS/CFT & Wilson loops III

Coupling constant in GR $\rightarrow G_N$

$$S_{EH} = \frac{1}{16\pi G_N}\int d^dx\,\sqrt{g}\,\underbrace{R[g]}_{\partial^2 g} \qquad \partial^2 g \rightarrow [R] = L^{-2}$$

$$[S] = L^0 \rightarrow \frac{[R]}{[G_N]} = L^{-d} \rightarrow \boxed{[G_N] = L^{-(2-d)}}$$

$\rightarrow [G_N] = L^{d-2}$

$$\boxed{G_N \sim (l_p)^{d-2}}$$

G_N defines a length scale a.k.a. Planck length

- G_N is dimensionless in $d=2$
- G_N is irrelevant for $d>2$

$$[G_N] = E^{-\alpha} \quad \alpha > 0\,!$$

Cosmological constant:

$$S_{EH} = \frac{1}{16\pi G_N}\int d^dx\,\sqrt{g}\,(R-\Lambda)$$

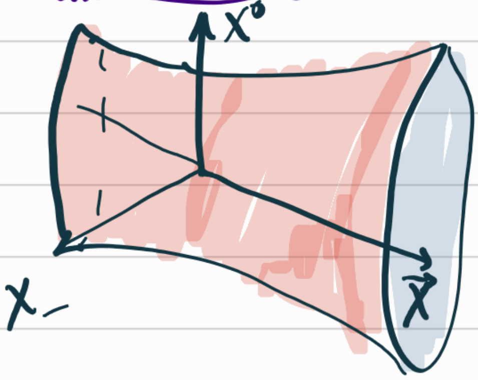
$\rightarrow [R] = [\Lambda] = L^{-2}$

Cosmological constant defines a characteristic length l_Λ

In the presence of Λ we can build a dimensionless quantity

$$\boxed{\left(\frac{l_\Lambda}{l_{pl}}\right)^{\#} \sim \frac{1}{\hbar}}$$

AdS_{d+1} geometry: can be visualized as a hypersurface in $\mathbb{R}^{2,d}$



X^i, $i=1,\dots d$

$$(X^0)^2+(X^-)^2-(\vec{X})^2 = \ell_{AdS}^2$$

inside $\underbrace{ds^2 = (dX^0)^2+(dX^-)^2-(d\vec{X})^2}_{\text{Naturally Manifest } SO(2,d)}$

linearly realized $X^A \to X'^A = \Lambda^A{}_B X^B$, $\Lambda \in SO(2,d)$

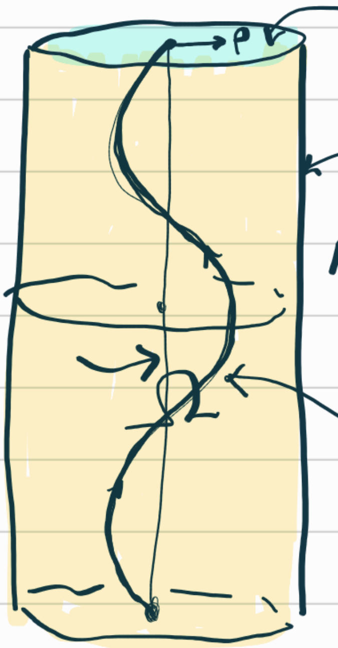
Particular solutions of the constraint lead to AdS in ≠ Coord Syst

eg: $X^0 = \ell_{AdS}\, ch\rho \cos t$, $X^i = \ell_{AdS}\, sh\rho\, n^i$ with $\vec{n}^2 = 1 \to S^{d-1}$ sphere

$X^- = \ell_{AdS}\, ch\rho \sin t$

this sln leads to AdS in global coords

$$ds^2 = \ell_{AdS}^2\left(-ch^2\rho\, dt^2 + d\rho^2 + sh^2\rho\, d\Omega_{d-1}\right)$$



the bulk $0 \leq \rho < \infty$

the boundary $\rho = \infty$

t

massive geodesic = they cannot reach the boundary

AdS in a box:

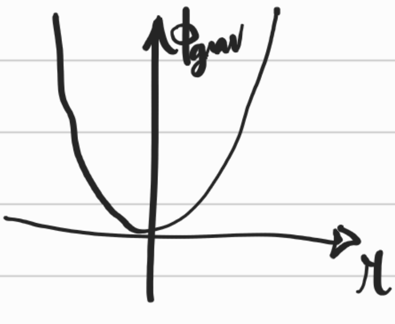
$$ds^2 = -\left(1+\left(\frac{r}{l_{AdS}}\right)^2\right)dt^2 + \frac{dr^2}{\left(1+\left(\frac{r}{l_{AdS}}\right)^2\right)} + r^2\, d\Omega_{d-1}$$

Recall that in the weak gravity limit we wrote in GR course

$$ds^2 = -(1+2\phi)\,dt^2 + \dots$$

↓ gravity potential eg. $2\phi = -\frac{2G_N M}{r}$ (Schwarzschild)

$\Rightarrow$ $\boxed{\phi_{grav} = \frac{1}{2}\frac{r^2}{l_{AdS}^2}}$ harmonic oscillator confining particles to the interior



The statement is in fact precisely more than an analogy the spectrum in the bulk is equispaced.

Particle in AdS. KG eqn: look for normal modes in AdS

$$\boxed{(\Box - m^2)\phi = 0}$$

natural ansatz

$$\phi(t,\rho,\Omega) = e^{-i\omega t}\, Y_{lm}(\Omega)\, f(\rho)$$

$$\Box = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\, g^{\mu\nu}\partial_\nu\right)$$

$$-\nabla^2_{S^n} Y_{lm} = l(l+n-1)\, Y_{lm}$$

The eqn for $f(\rho)$ turns out to be

$$\frac{1}{ch\rho(sh\rho)^{d-1}}\frac{d}{d\rho}\left((sh\rho)^{d-1}ch\rho\,\frac{df}{d\rho}\right)+\left(\frac{\omega^2}{ch^2\rho}+\frac{q^2}{sh^2\rho}-(m\ell_{AdS})^2\right)f(\rho)=0$$

$q=\ell(\ell+d-2)$

This 2^{nd} order eqn will give two slns

$$f(\rho)=C_1\,f_1(\rho)+C_2\,\cancel{f_2(\rho)}\quad,$$ f_1 & f_2 are hypergeometrics $_2F_1$

- Asking for regularity at $\rho=0$ elliminates C_2 piece
- Imposing normalizability at $\rho=\infty$ quantizes ω!

for the regular sln we obtain

$$f(\rho)=\frac{(th\rho)^{\ell}}{(ch\rho)^{\Delta_+}}\;{}_2F_1\left(\tfrac{1}{2}(\Delta_++\ell-\omega),\tfrac{1}{2}(\Delta_++\ell+\omega),\tfrac{d}{2}+\ell,th^2\rho\right)$$

where $\boxed{\Delta(\Delta-d)=(m\ell_{AdS})^2}\to\Delta_\pm=\frac{d}{2}\pm\sqrt{\left(\frac{d}{2}\right)^2+(m\ell_{AdS})^2}$

the asymptotic behavior at ∞ gives

$$f(\rho)\sim \#\,e^{-\Delta_+\rho}+\#\,e^{-\Delta_-\rho}$$

notice $(m\ell_{AdS})^2>0\Rightarrow \Delta_+>0$, $\Delta_-<0$

not allowed

this coef takes the form $\dfrac{\Gamma\ \cdots}{\Gamma\left(\underbrace{\tfrac{1}{2}(\Delta_++\ell-\omega)}_{-n}\right)}$

if we impose

$$\boxed{\omega_n=\Delta_++\ell+2n}\quad\Longleftarrow$$ the dangerous decay vanishes

Quantized energy in Global AdS as harmonic osc.

It is now straightforward to quantize a KG field in AdS

$$\phi(t,\rho,\Omega) = \sum_{nlm} q_{nlm}\, e^{i\omega_n t}\, Y_{lm}(\Omega)\, f_{nlm}(\rho) + h.c.$$

$$\downarrow$$

$$\hat{\phi} = \sum_{nlm} \left(a_{nlm}\, e^{-i\omega_n t} \cdots + a^\dagger_{nlm}\, e^{i\omega_n t} \cdots \right)$$

and we construct the Fock space as

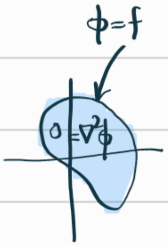
$$a_{nlm} |0\rangle = 0 \quad \forall n, l, m$$

1-particle states are defined as

$$\boxed{|n,l,m\rangle \equiv a^\dagger_{nlm} |0\rangle}$$

Boundary conditions in AdS

Consider an electro problem (now in Euclidean signat)



$$-\nabla^2 \phi = 0$$

$$\phi\big|_{\partial AdS} = f(\vec{x})$$

bdy coords

Alternatively we can solve this problem in Poincaré coords

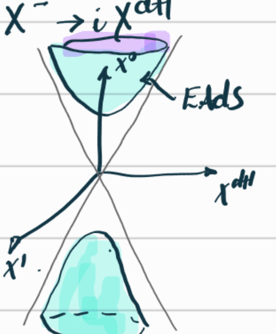
$$\boxed{ds^2 = l_{AdS}^2 \frac{(dz^2 + d\vec{x}^2)}{z^2}}$$

Boundary locates at $z = 0$.

In Euclidean space we make $X^- \to i X^{d+1}$ the quadric turns into

$$(X^0)^2 - (X^m)^2 = l_{AdS}^2$$

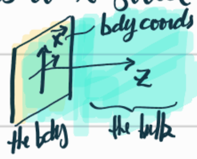
$$m = 1, \ldots d+1$$



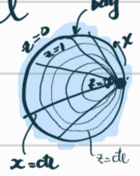
Solving the constraint as before

$$ds^2 = l_{AdS}^2 \left(d\rho^2 + \sinh^2\rho\, d\Omega \right)$$

We can picture EAdS as a solid ball



in 2d



The boundary problem at ∞ in AdS is well posed in AdS (Fefferman+Graham)

Again we solve (in Euclidean space) for modes in EAdS, the solution near the boundary can be found with ansatz $\phi(z,\vec{x}) = e^{i\vec{p}\cdot\vec{x}} F(z)$

$$(-\nabla^2 + m^2)\,\phi(z,\vec{x}) = 0$$

$$\downarrow$$

$$z^2 F'' + (1-d) z F' + (-m^2 - \vec{p}^{\,2} z^2) F = 0$$

Near the boundary $z=0$ we look for $F(z) = z^{\Delta}$ then Δ must satisfy

$$\Delta(\Delta - d) = m^2 \qquad (\ell_{AdS} = 1)$$

$$\hookrightarrow \Delta_{\pm} = \frac{d}{2} \pm \sqrt{\left(\frac{d}{2}\right)^2 + (m\,\ell_{AdS})^2}$$

$\hookrightarrow$ the ones that appeared before

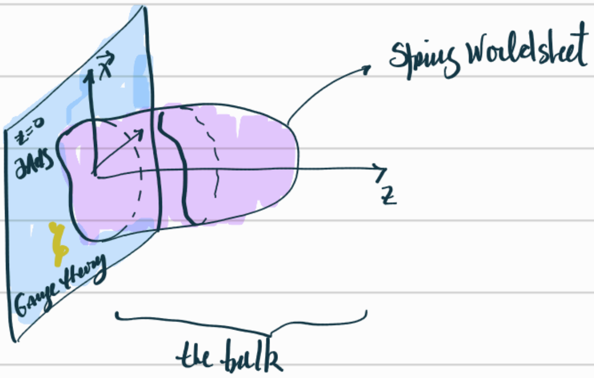
This gives us an interpretation to $\Delta_{\pm}$: they are the decay behavior of the field at the AdS boundary. Generically

$$F(z) = \overset{\text{source}}{C_-}\, z^{\Delta_-}(1 + a_1 z + \dots) + \overset{\text{VEV/response}}{C_+}\, z^{\Delta_+}(1 + \dots)$$

When we talk about setting bdy conditions @ ∂AdS we think about setting the "non-normalizable" piece of ϕ, i.e. $C_- = C_-(\vec{p})$. Regularity of the solution in the deep interior $z = \infty$ will fix $\boxed{C_+ = C_+(C_-)}$.

"the vev" "the source"

Wilson loops in AdS/CFT



$$\langle W[\gamma]\rangle = \int_{\partial X^\mu = \gamma} \mathcal{D}X^\mu \mathcal{D}h \, e^{-S_{pol}[X]}$$

$$\approx e^{-\text{Area}(X_{cl}) + \text{corrections}}$$

The relevant parameter controlling this computation is

$$\boxed{T_{eff} = \frac{l_{AdS}^2}{\alpha'} = \sqrt{\lambda}}$$

Corrections to this result behave as $\left(\frac{1}{\sqrt{\lambda}}\right)^n$ and follow from fluctuations around the classical worldsheet X^μ_{cl}. Concretely we have

$$X^\mu(\sigma) = X^\mu_{cl}(\sigma) + \underbrace{y^\mu}_{\text{fluctuations}}$$

$$S_{pol}[X] = \sqrt{\lambda} \int d^2\sigma \sqrt{h}\, h^{\alpha\beta} G_{\mu\nu}(X)\, \partial_\alpha X^\mu \partial_\beta X^\nu + \dots$$

$$\simeq S_{pol}[X^\mu_{cl}] + \cancel{\frac{\delta S_{pol}}{\delta X^\mu}\Big|_{X_{cl}}} y^\mu + \frac{1}{2} y^\mu \frac{\delta^2 S}{\delta X^\mu \delta X^\nu} y^\mu + \dots$$

since I am evaluating on the classical sln

Hence

$$\int \mathcal{D}X^\mu e^{-S_{pol}} \simeq \int dy^\mu \, e^{-S_{pol}[X_{cl}]} \, e^{-\frac{1}{2}\int y \cdot \frac{\delta^2 S}{\delta y^2}\cdot y}$$

where $\mathcal{O} = \frac{\delta^2 S}{\delta y^2}$,

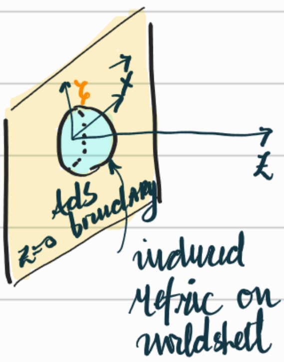
$$\simeq e^{-S_{pol}[X_{cl}]} \frac{1}{\det^{1/2}\mathcal{O}}$$

$$\simeq e^{\underbrace{-\sqrt{\lambda}\,\mathrm{Area}(X_{cl})}_{\text{leading } \sqrt{\lambda}} \underbrace{-\frac{1}{2}\log\det\mathcal{O}}_{\text{subleading } (\sqrt{\lambda})^0} + O\left(\frac{1}{\sqrt{\lambda}}\right)}$$

← Caveat: Zero Modes bring $\log\sqrt{\lambda}$ contributions (see Coleman Instant Lect)

This computation can be contrasted with the localization result (Pestun) coming from field theory.

Paradigm: circular WL ⇝ start with Nambu+GoTo action



$$S_{NG} = \frac{\sqrt{\lambda}}{2\pi}\int d^2\sigma \sqrt{g}$$ ← induced worldsheet metric

$$g_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \underbrace{G_{\mu\nu}(X)}_{AdS_5\times S^5} = \begin{pmatrix} \dot{X}^2 & \dot{X}\cdot X' \\ \dot{X}\cdot X' & X'^2 \end{pmatrix}$$

$$ds^2 = \frac{+dt_E^2 + dz^2 + dr^2 + r^2 d\phi^2 + dy^2}{z} + d\Omega_5^2$$

In this bkd background we consider the ansatz

$$X^M(\sigma,\tau) = \left(t_E = \text{cte}, z(\sigma), r(\sigma), \phi = \tau, y = \text{cte}, \Omega = \text{cte}\right)$$

$\dot{X}^M = \delta^M_\phi$

$X'^M = z'\delta^M_z + r'\delta^M_r$

Inserting in NG Action we find

$$S_{NG} = \frac{\sqrt{\lambda}}{2\pi}\int d^2\sigma \sqrt{\underbrace{\dot{X}^2}_{r^2/z^2} X'^2 - \cancel{(\dot{X}\cdot X')^2}} = \frac{\sqrt{\lambda}}{2\pi}\int \frac{d\sigma\, d\tau\, r(\sigma)}{z^2(\sigma)}\sqrt{z'^2 + r'^2}$$

It remains to fix σ-diffeo, the sln results:

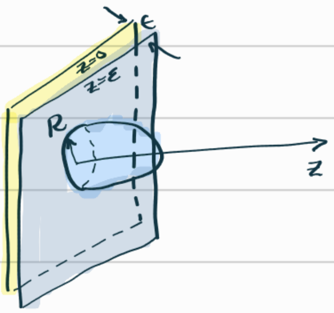
tip boundary

$$r(\sigma)=\sigma \qquad z(\sigma)=\sqrt{R^2-\sigma^2}\,, \qquad \tau\in(0,2\pi) \qquad \sigma\in(0,R)$$

R is an integration constant fixing the radius of the circle at the boundary C. What about conformal invariance?

<u>On shell action</u>: plugging in ⊛ we find ∞ due to the ∞ extension of AdS $\rightsquigarrow$ we regularize by placing C at $z=\varepsilon$, then

$$S_{NG}=\sqrt{\lambda}\int_0^{\sqrt{R^2-\varepsilon^2}} \frac{dr\, r}{z^2(r)}\sqrt{1+z'^2} = \underbrace{\underbrace{\sqrt{\lambda}\,\frac{R}{\epsilon}}_{AdS_2\ \text{"}\infty\text{" Volume}}}_{\substack{\text{proportional to}\\ \text{loop's length can be elliminated by } e^{-\Lambda\oint ds}\ C\ \text{Counterterm}}} - \underbrace{\sqrt{\lambda}}_{\substack{\text{Regularized}\\ AdS_2\ \text{Vol}}} + O(\varepsilon)$$



$$\boxed{\mathrm{Vol}(AdS_2) = -2\pi}$$

$$\Rightarrow \quad \langle W[C]\rangle \sim e^{-S_{pol}[X_{cl}]} \sim e^{\sqrt{\lambda}}$$

We can envisage computing "semiclassical" corrections as described in the previous page. As an outcome one finds agreement with the localization result expanded at strong coupling.

Fin!